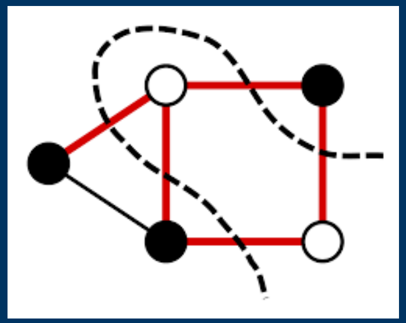
Approximation Algorithms for Noncommutative Constraint Satisfaction Problems

Maximize $< \phi | A_0B_0 + A_0B_1 + A_1B_0 - A_1B_1 | \phi >$



Max-Cut

Hamoon Mousavi (Simons at UC Berkeley)

Eric Culf (University of Waterloo), Taro Spirig (University of Copenhagen)

Magic Square

$x_{ij} \in \{+1, -1\}$

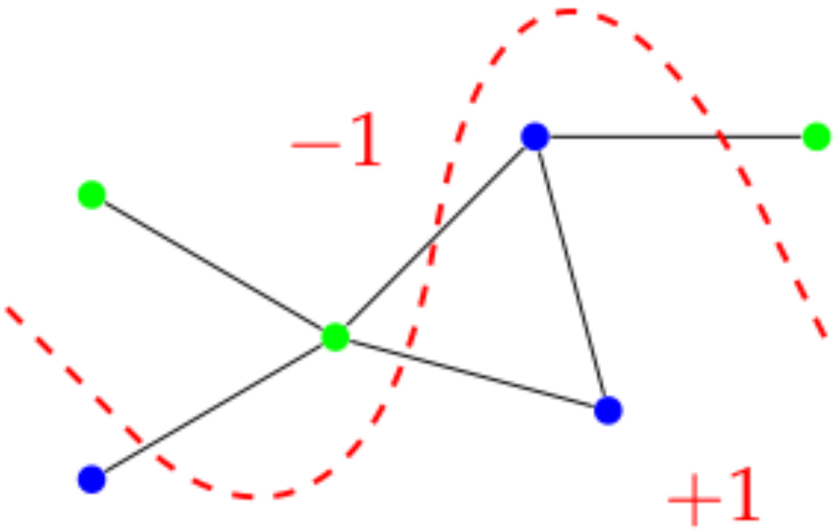
x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

Operator Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

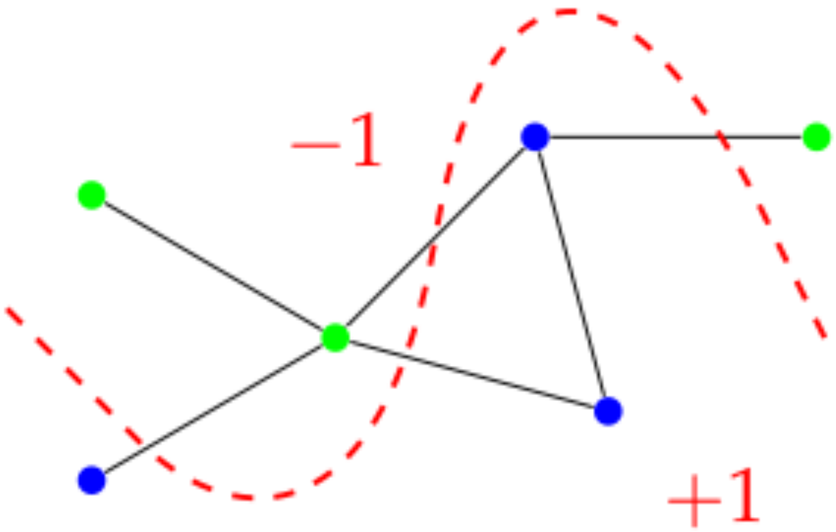
$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

Noncommutative Max-Cut

$$\max \quad \sum \frac{1 - X_i X_j}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

Noncommutative Max-Cut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

Noncommutative Max-Cut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

- The Hilbert space is finite-dimensional
- But no bound on the dimension
- tr is the dimension-normalized trace
- $tr(XY)$ is always between -1 and 1

$$tr(XY) = \langle\psi|(XY \otimes I)|\psi\rangle$$

Where $|\psi\rangle$ is a maximally entangled state on a larger system

Probabilistic Cut: an assignment of binary random variables

Probabilistic Cut: an assignment of binary random variables

$Pr_1(+1) = 0.8$ $\quad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad$ $Pr_2(-1) = 0.5$

Probabilistic Cut: an assignment of binary random variables

$Pr_1(+1) = 0.8$ $\quad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad$ $Pr_2(-1) = 0.5$

This then induces a probability distribution over cuts

A probabilistic cut: An ensemble of cuts

Probabilistic Cut: an assignment of binary random variables

$Pr_1(+1) = 0.8$ $\quad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad$ $Pr_2(-1) = 0.5$

Noncommutative Cut: an assignment of binary observables

Probabilistic Cut: an assignment of binary random variables

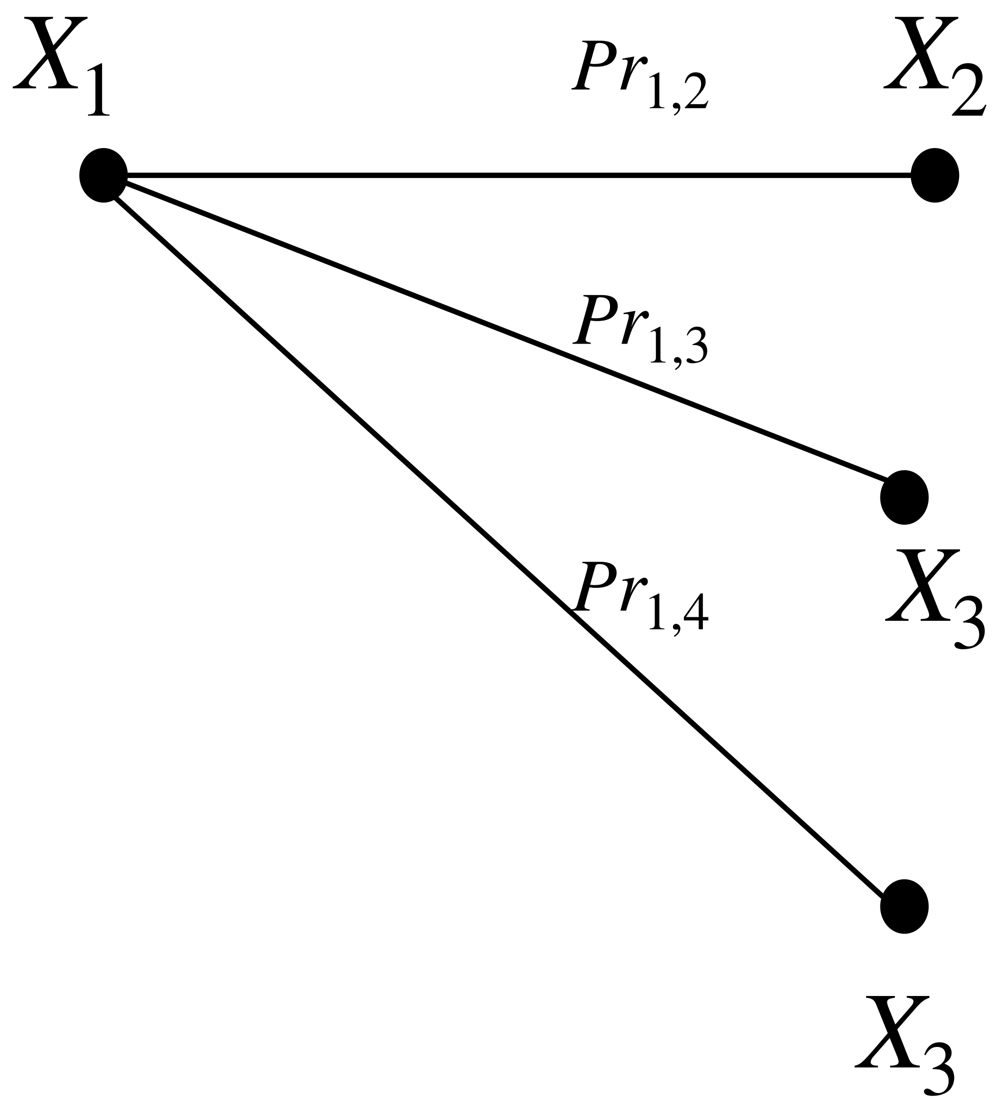
$Pr_1(+1) = 0.8$ $\quad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad$ $Pr_2(-1) = 0.5$

Noncommutative Cut: an assignment of binary observables

$$Pr_{12}(+1, +1) = 0.1$$
$$Pr_{12}(+1, -1) = 0.2$$
$$Pr_{12}(-1, +1) = 0.3$$
$$Pr_{12}(-1, -1) = 0.4$$

Inconsistencies of Edge Probabilities



Probabilistic Cut

$Pr_1(+1) = 0.8$ $\qquad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\qquad$ $Pr_2(-1) = 0.5$

Noncommutative Cut

$$Pr_{12}(+1, +1) = 0.1$$
$$Pr_{12}(+1, -1) = 0.2$$
$$Pr_{12}(-1, +1) = 0.3$$
$$Pr_{12}(-1, -1) = 0.4$$

Noncommutative Cut

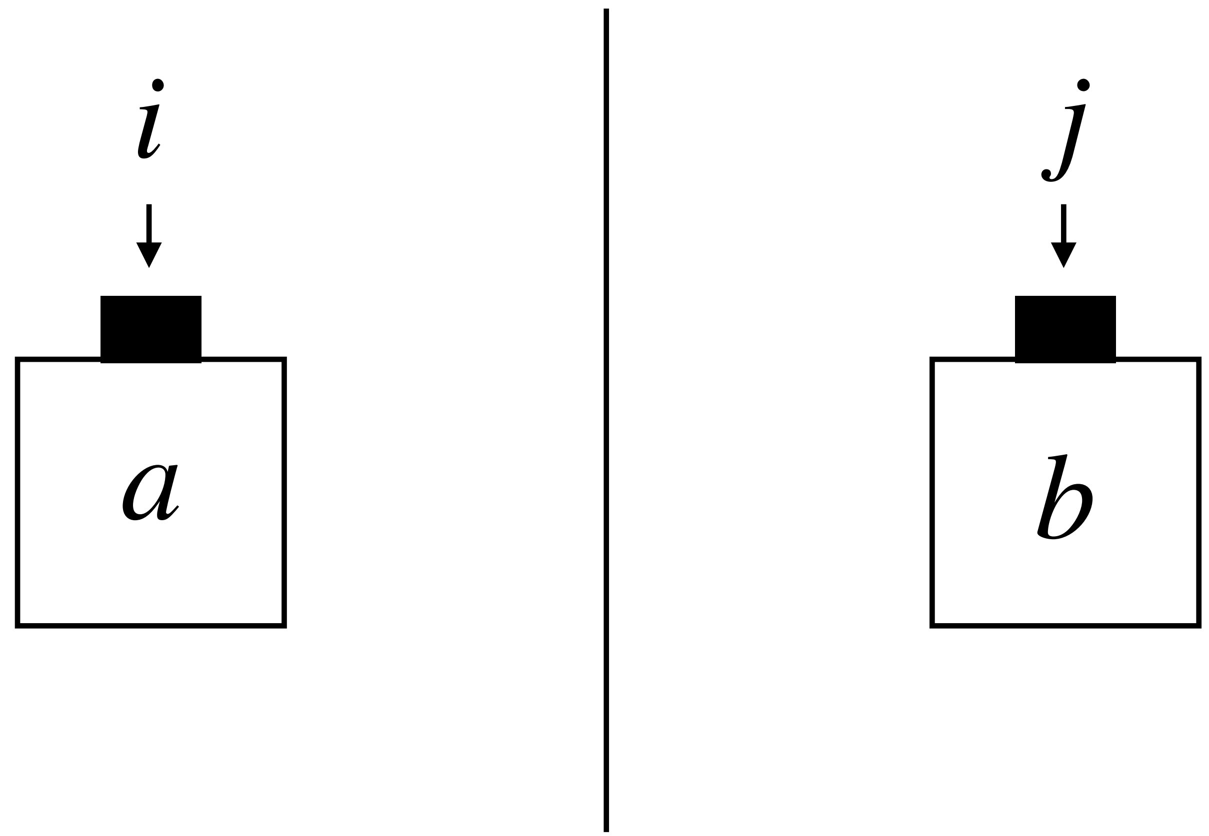
$$Pr_{12}(+1, +1) = tr(\frac{I + X_1}{2}\frac{I + X_2}{2})$$

$$Pr_{12}(+1, -1) = tr(\frac{I + X_1}{2}\frac{I - X_2}{2})$$

$$Pr_{12}(-1, +1) = tr(\frac{I - X_1}{2}\frac{I + X_2}{2})$$

$$Pr_{12}(-1, -1) = tr(\frac{I - X_1}{2}\frac{I - X_2}{2})$$

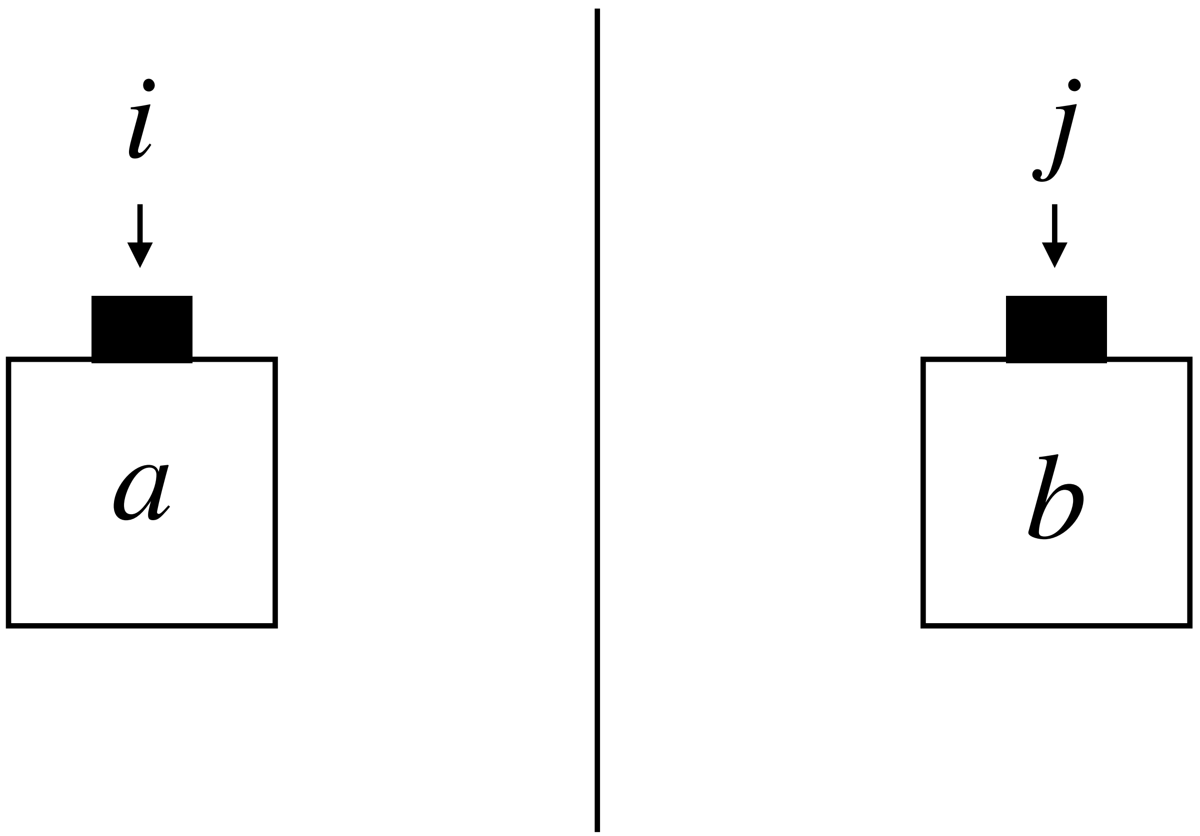
Operational Interpretation of Noncommutative Cuts



$i, j \in V,$

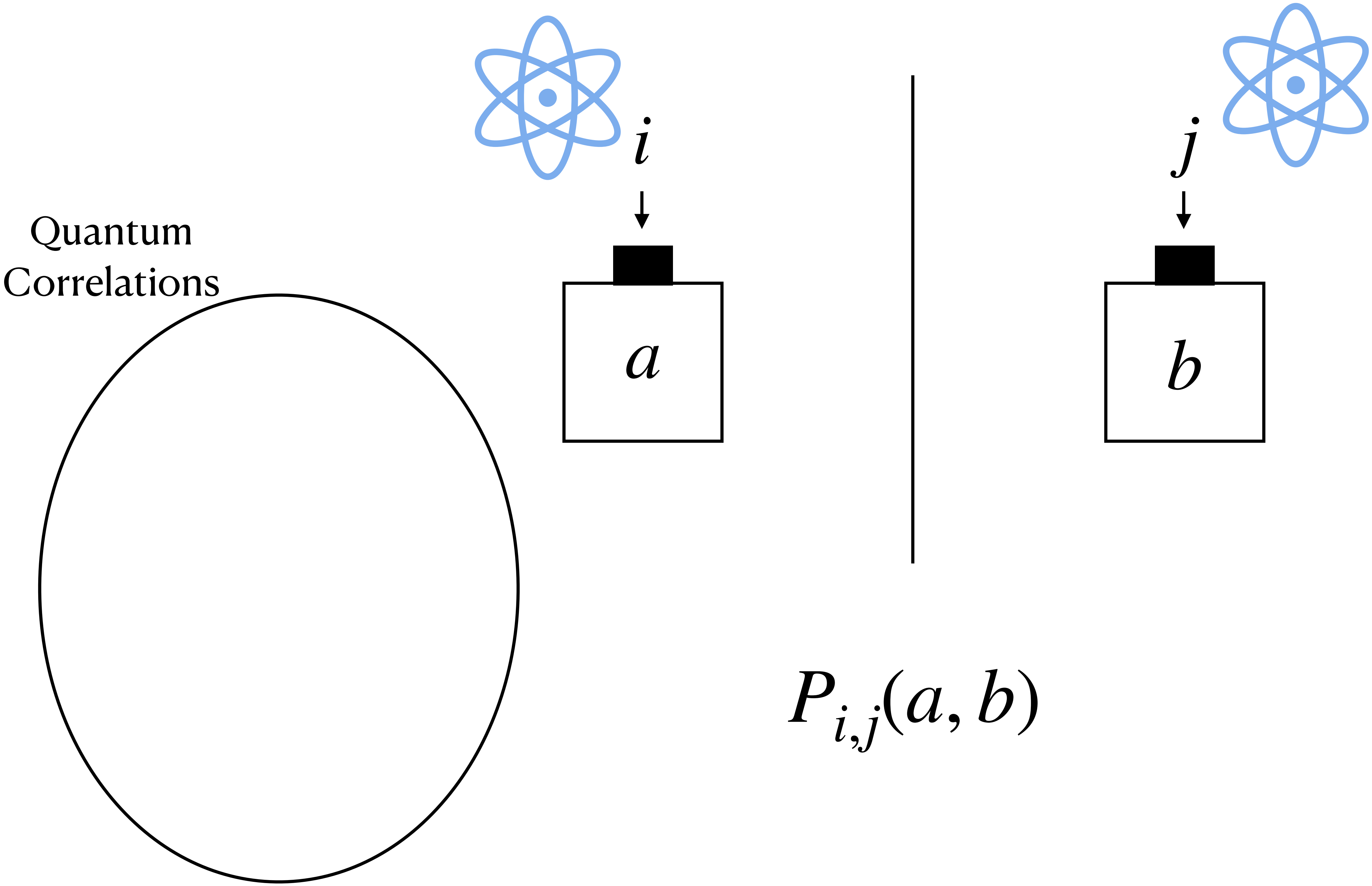
$a, b \in \{+1, -1\}$

Correlations

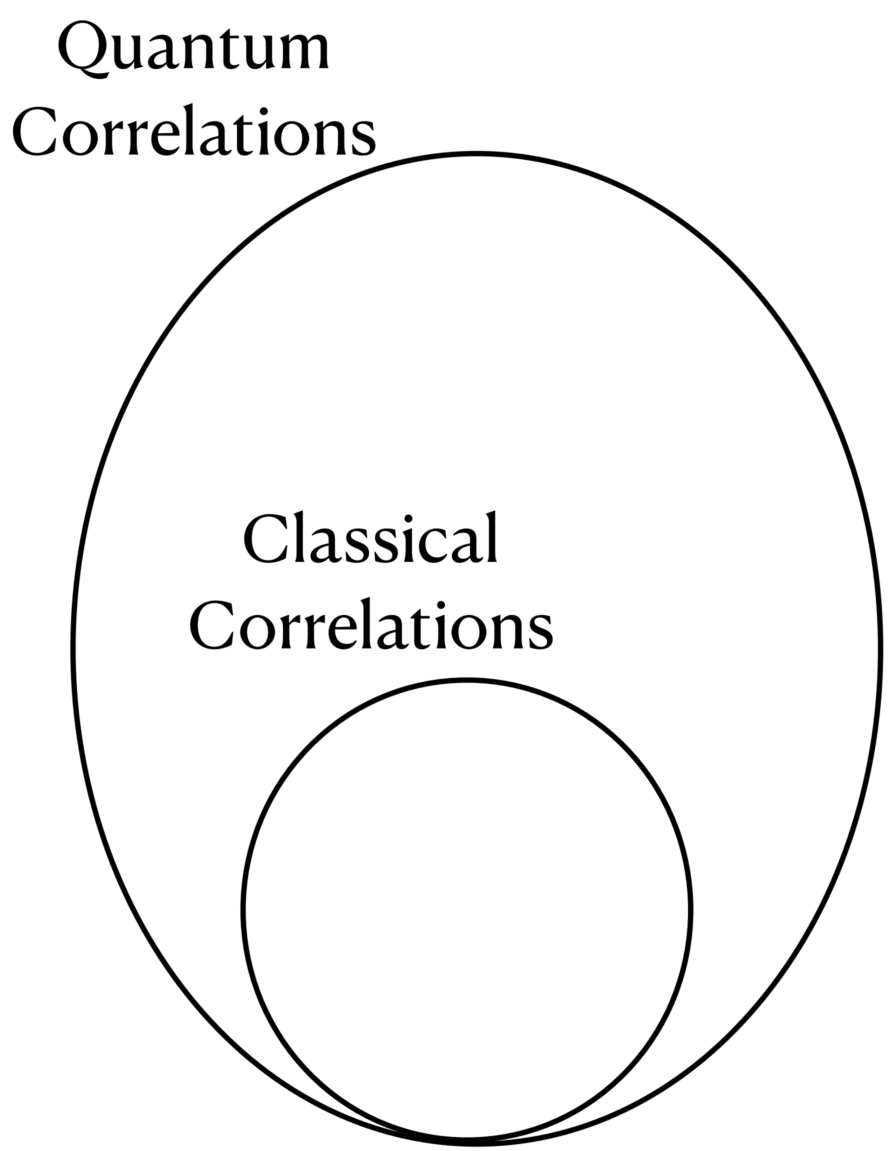


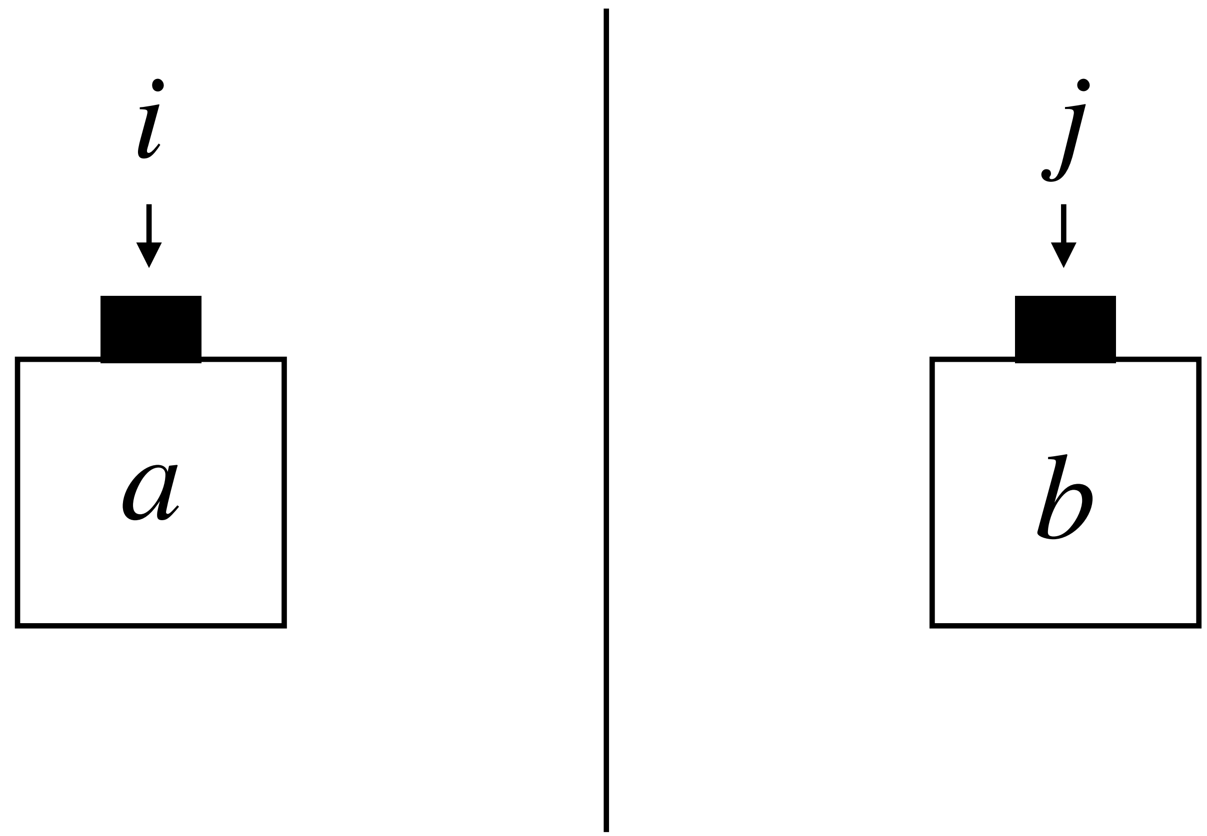
$$P_{i,j}(a, b)$$

Quantum Correlations



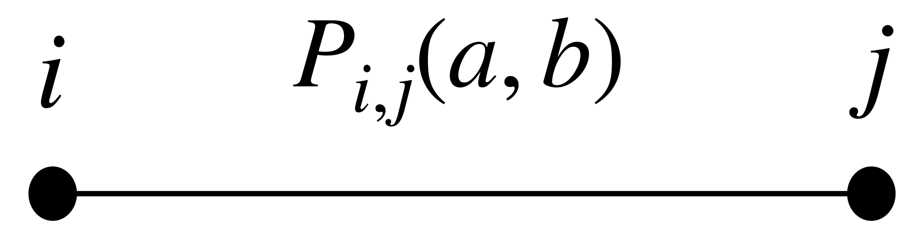
Classical Correlations





$$P_{i,j}(a, b)$$

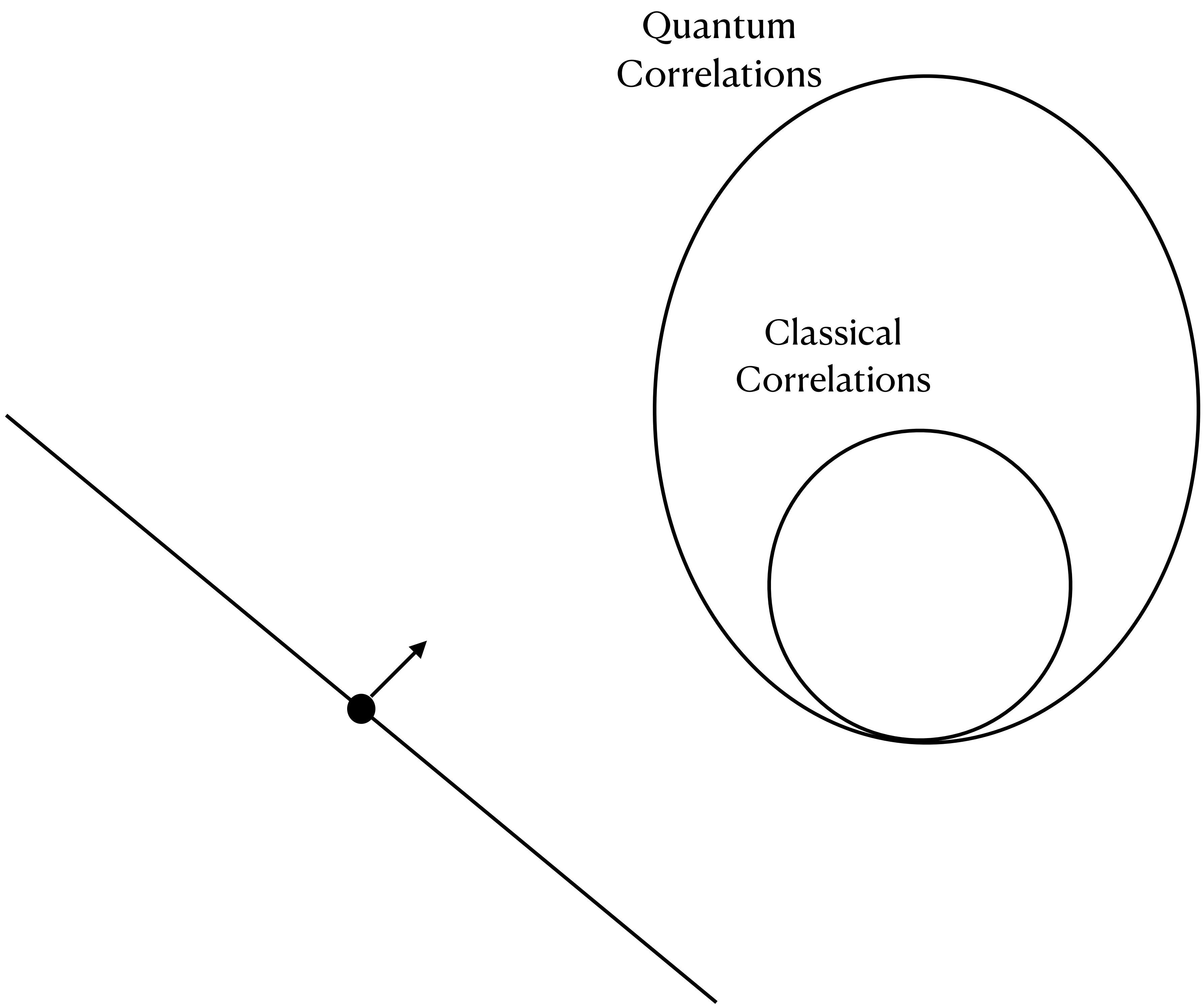
Edge Probabilities



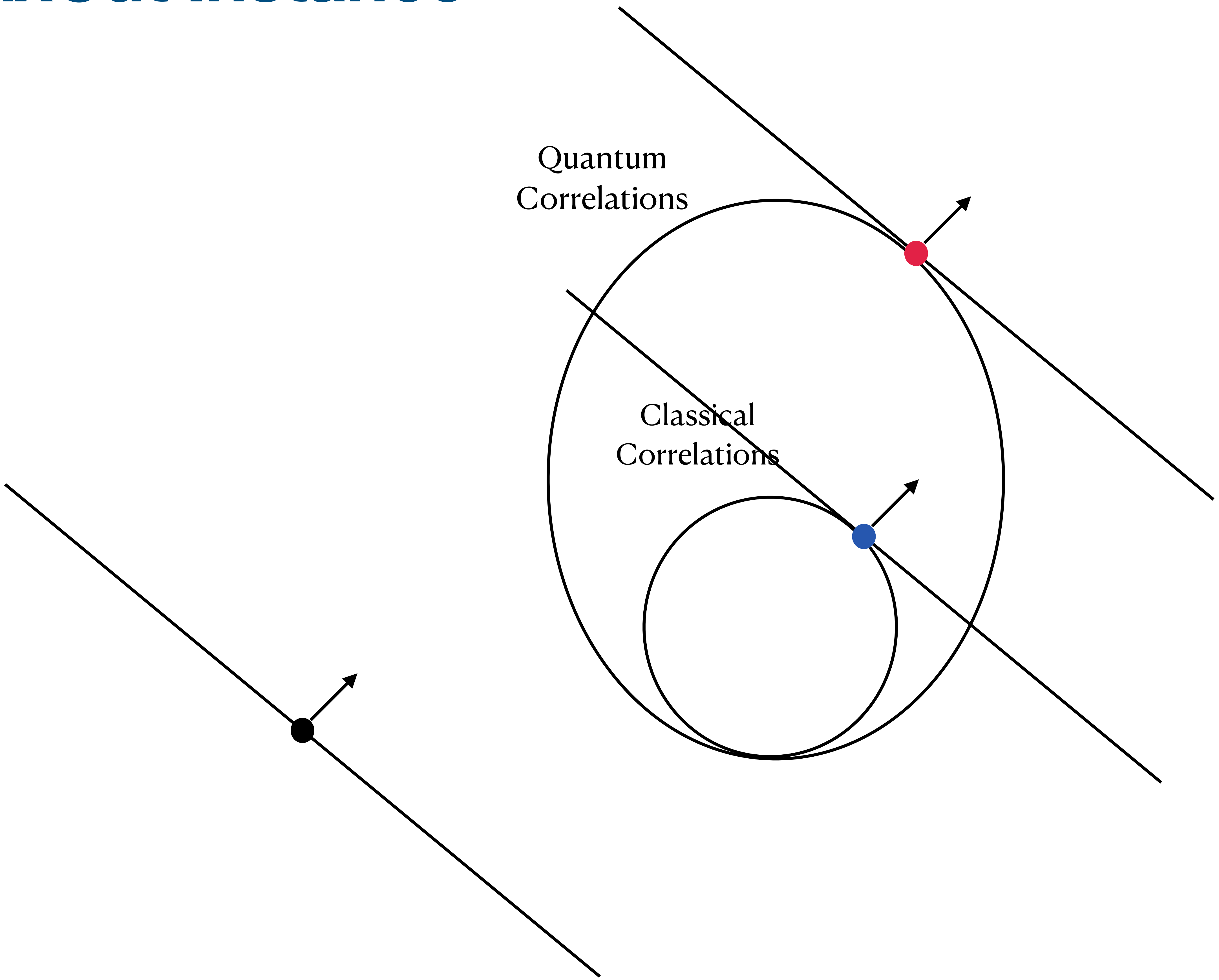
Quantum Correlations = Noncommutative Cuts

Classical Correlations = Probabilistic Cuts

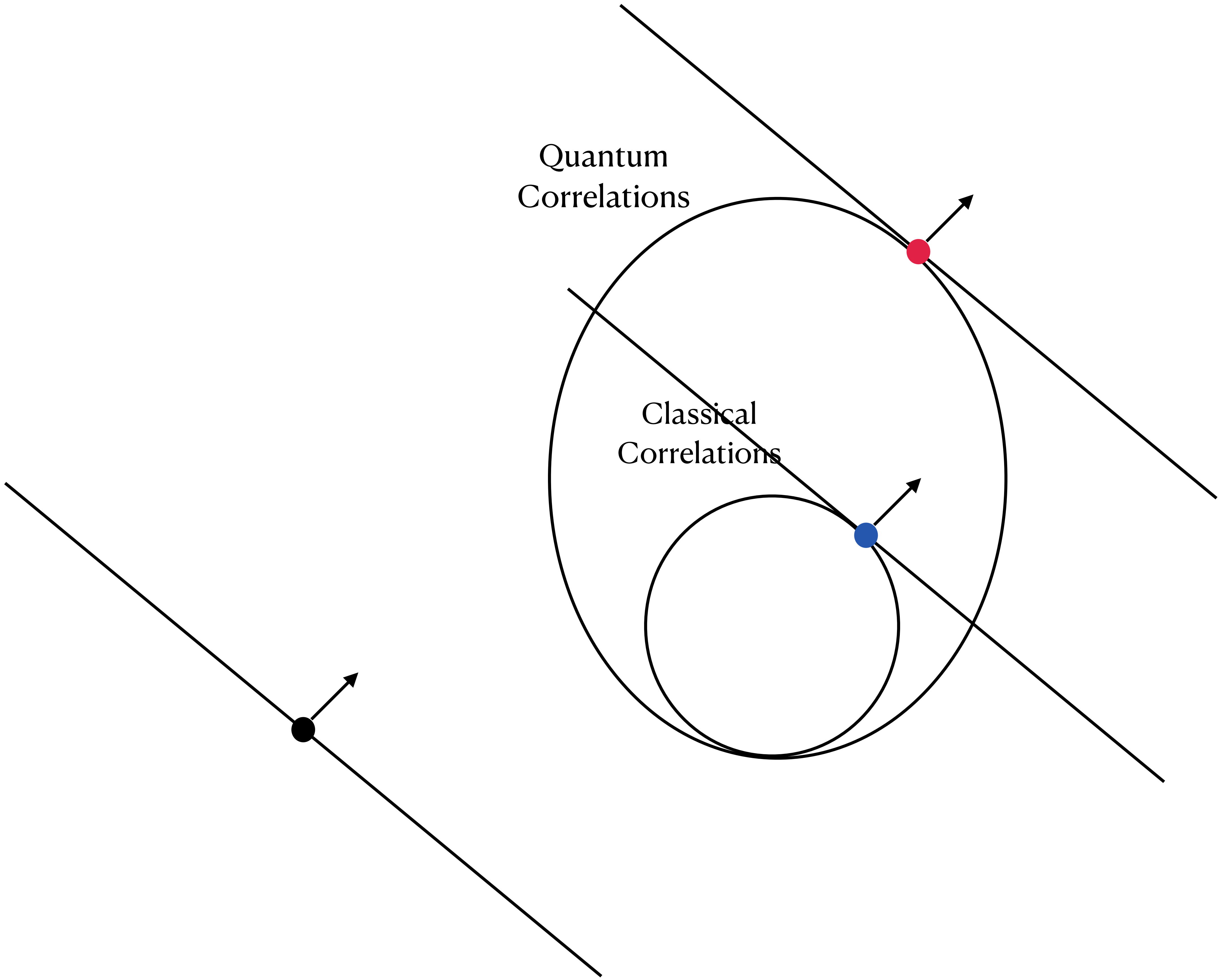
MaxCut Instance



MaxCut Instance



The 2022 Nobel Prize in Physics awarded to Alain Aspect, John F. Clauser, and Anton Zeilinger



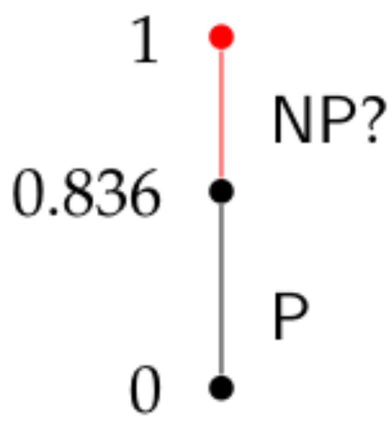
Computational Aspects

- Slofstra 2016: Membership problem for "Quantum Correlations" is undecidable
- In particular optimization over the set is uncomputable
- Ji, Natarajan, Vidick, Wright, Yuen 2020: Approximation is also beyond reach
- Tsirelson 1980: Noncommutative MaxCut is in P
- Karp 1972: Classical MaxCut is NP-Complete

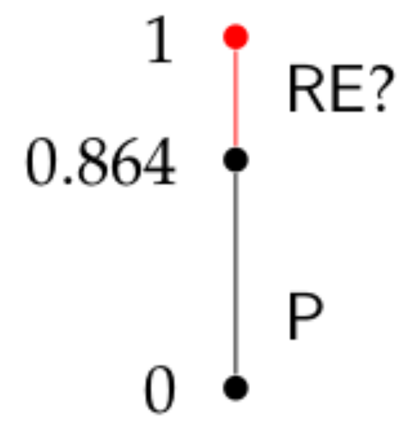
Computational Aspects

- Slofstra 2016: Membership problem for "Quantum Correlations" is undecidable
- In particular optimization over the set is uncomputable
- Ji, Natarajan, Vidick, Wright, Yuen 2020: Approximation is also beyond reach
- Tsirelson 1980: Noncommutative MaxCut is in P
- Karp 1972: Classical MaxCut is NP-Complete
- Classical theory: General CSPs are NP-hard to approximate, but what about special cases like MaxCut or Max3SAT?

Approximability of Noncommutative CSPs



(a) Max-3-Cut



(b) Noncommutative Max-3-Cut

Algorithm: Frieze and Jerrum
Goemans and Williamson
de Klerk, Pasechnik, and Warners

Hardness: Khot, Kindler, Mossel, O'Donnell

Concepts: Anticommuting Algebras and Relative Distributions

- Hyperplane rounding of Goemans-Williamson $\vec{r} = (r_1, \ldots, r_n)$
- A random operator $R = r_1\sigma_1 + \cdots + r_n\sigma_n$
- σ_i's generate generalized Weyl-Brauer algebra

Concepts: Anticommuting Algebras and Relative Distributions

- Given a λ, sample unitaries U, V uniformly such that $< U, V > = \lambda$
- Sample eigenvalues α, β from U, V
- What is the angle between α, β?
- It is the well-known Cauchy distribution