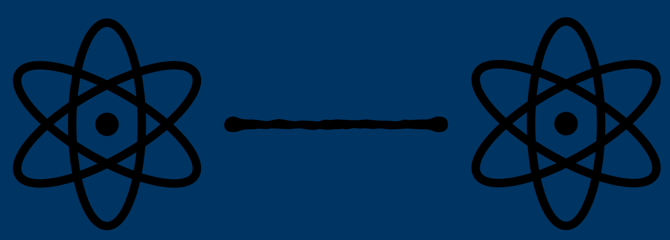
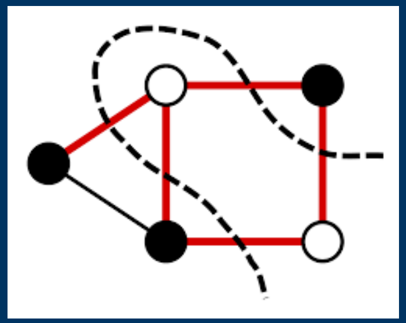
Algebras, CSPs, and Quantum Computation



Maximize $< \phi | A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 | \phi >$



Max-Cut

Hamoon Mousavi (Simons Institute at UC Berkeley)

Algebras, CSPs, and Quantum Computation

Plan

- Noncommutative constraint satisfaction problems (NC-CSPs)
    - Classical theory
    - Noncommutative extension
- New directions?
    - Quantum computation

Algebras, CSPs, and Quantum Computation

NC-CSP Terminology

- Quantum nonlocal games
- Bell inequalities
- Entangled multiprover interactive proofs (MIP*)
- Noncommutative polynomial optimization

- 3SAT instance is a boolean formula
- NC-3SAT should evoke a similar picture but for the quantum setting

NC-CSPs

Magic Square

$x_{ij} \in \{+1, -1\}$

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

Perfect Operator Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

$$x_{ij} \in \{+1, -1\}$$

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$x_{ij} \in \{+1, -1\}$

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Binary alphabet $\{+1, -1\}$ in the classical case $\longrightarrow$ Binary observables

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$x_{ij} \in \{+1, -1\}$

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Binary alphabet $\{+1, -1\}$ in the classical case $\longrightarrow$ Binary observables

Binary observables: Unitary operators with $\{+1, -1\}$ eigenvalues

$$O^*O = O^2 = I$$

An operator CSP

$X_{ij}^* X_{ij} = I$

$X_{ij}^2 = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

An operator CSP

$X_{ij}^* X_{ij} = I$

$X_{ij}^2 = I$

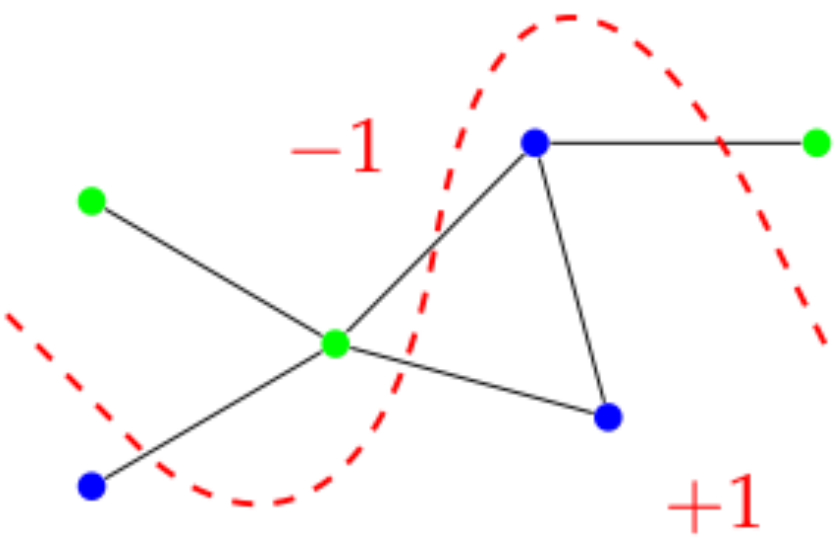
X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

When restricting to one dimension we recover the classical CSP

Because ± 1 are the only binary observables is one dimension

An operator CSP but not yet an NC-CSP

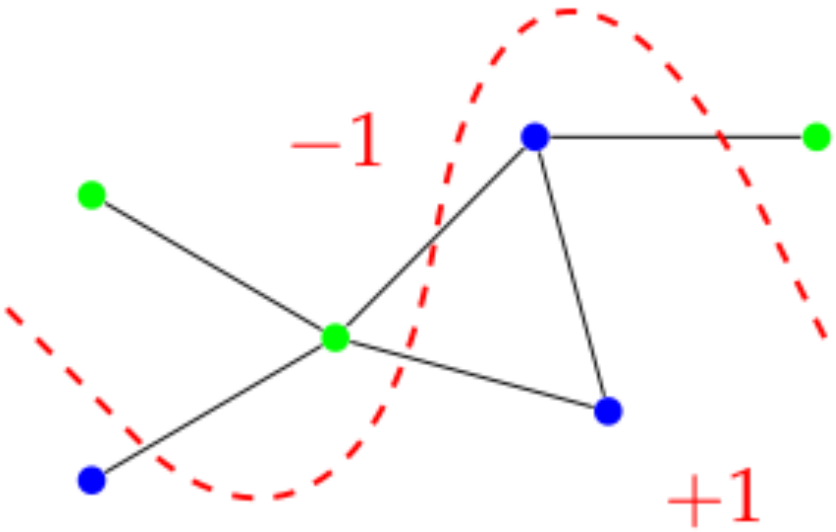
Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

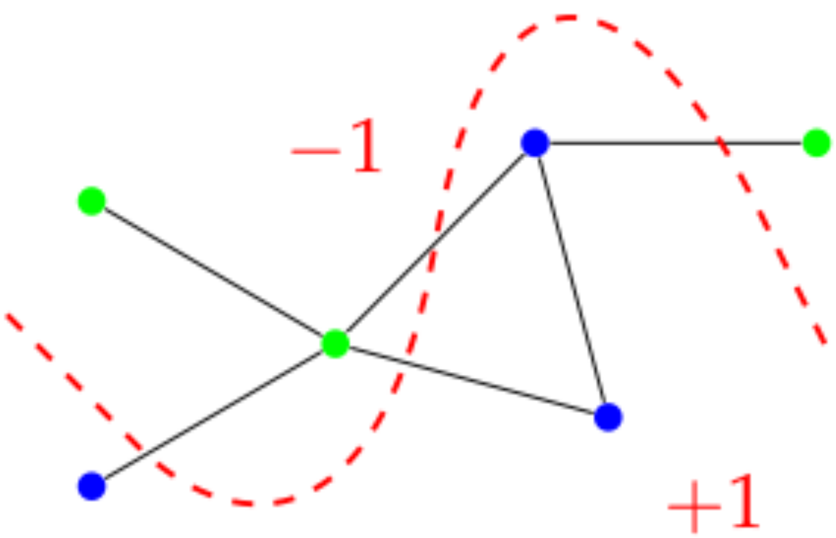
$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

NC-Max-Cut?

$$\max \quad \sum \frac{I - X_i X_j}{2}$$

s.t. X_i are binary observables

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

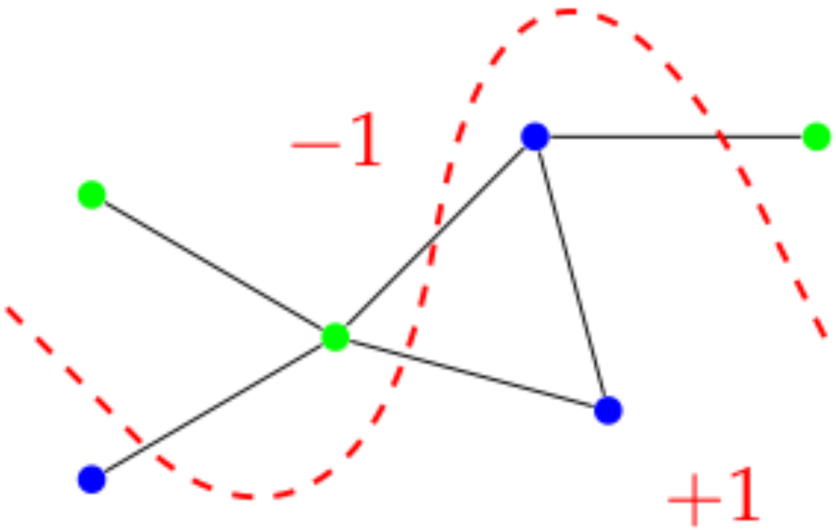
NC-Max-Cut?

$$\max \quad \langle \phi | \left(\sum \frac{I - X_i X_j}{2} \right) | \phi \rangle$$

s.t. X_i are binary observables

and $|\phi\rangle$ is a state

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

NC-Max-Cut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

Noncommutative Max-Cut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

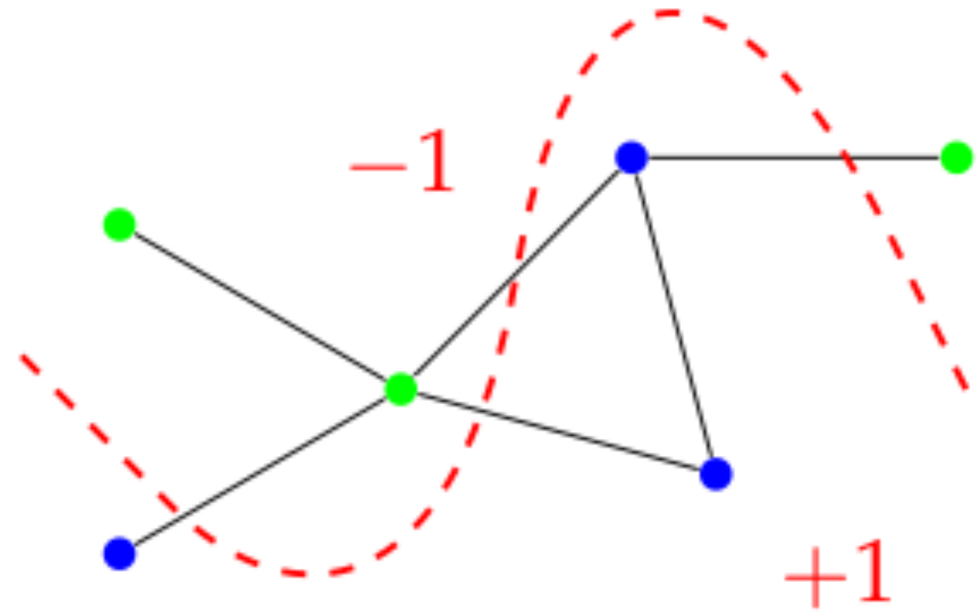
s.t. $\quad X_i \quad$ is unitary with ± 1 eigenvalues

- The Hilbert space is finite-dimensional
- But no bound on the dimension
- tr is the dimension-normalized trace
- $tr(XY)$ is always between -1 and 1

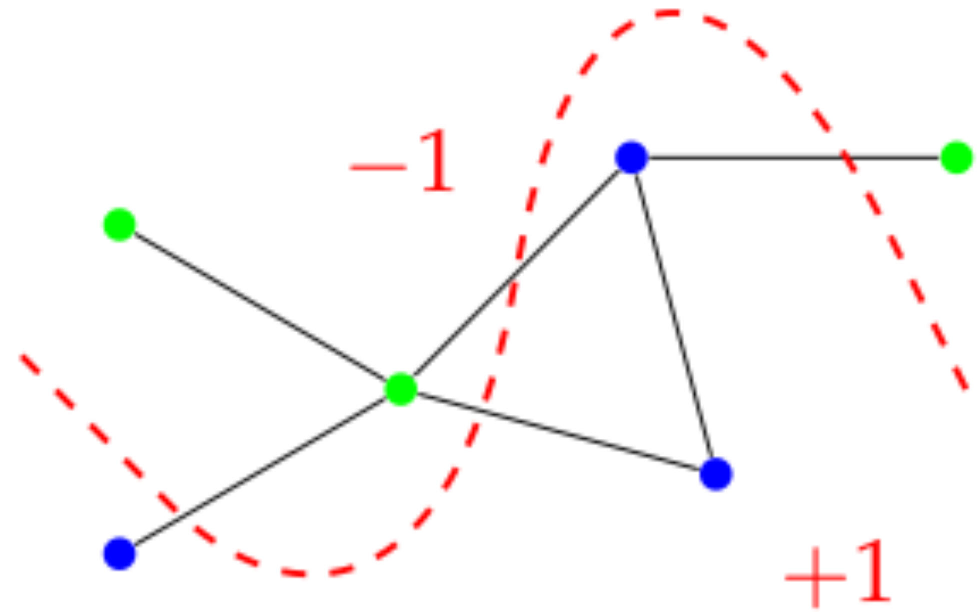
There is a state underlying the trace formulation

$$tr(XY) = \langle \psi | (XY \otimes I) | \psi \rangle$$

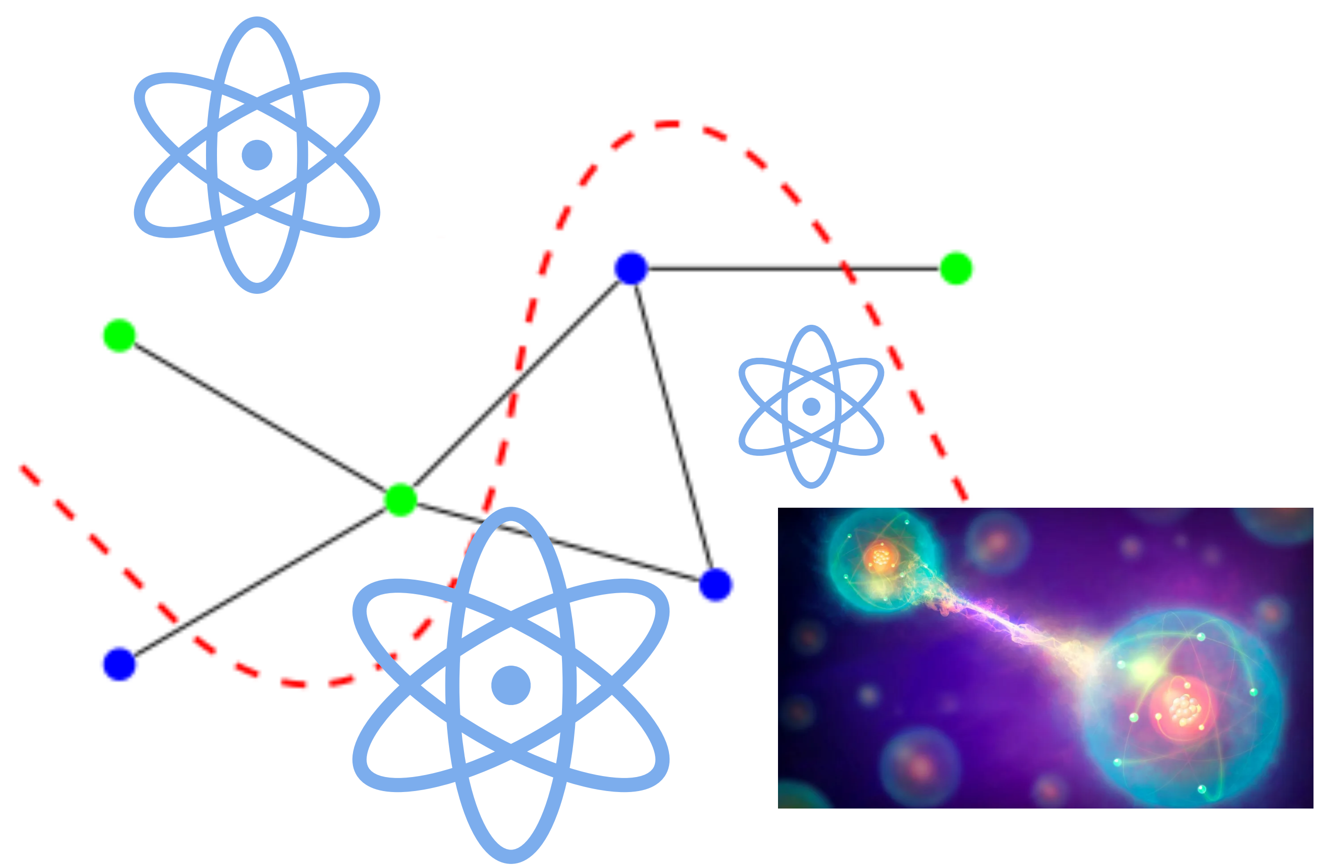
Where $|\psi\rangle$ is a maximally entangled state on a larger system



−1
+1



But what does a noncommutative cut look like?



Probabilistic Cut: an assignment of binary random variables

Probabilistic Cut: an assignment of binary random variables

$Pr_1(+1) = 0.8$ $\quad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad$ $Pr_2(-1) = 0.5$

Probabilistic Cut: an assignment of binary random variables

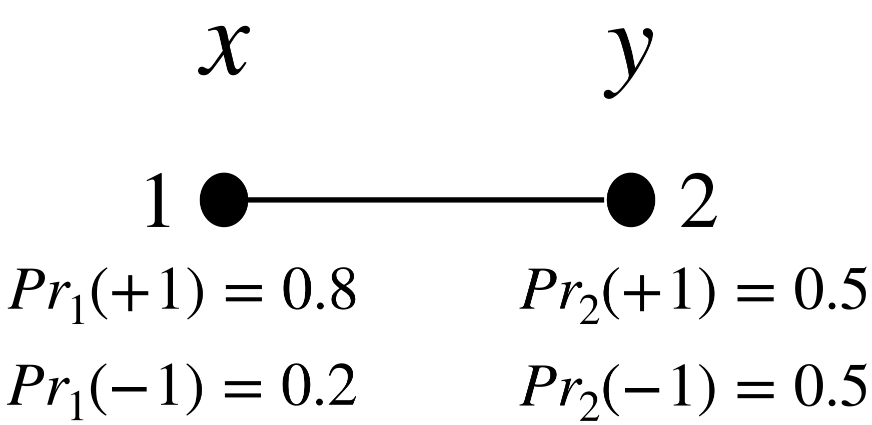
$Pr_1(+1) = 0.8$ $\quad Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad Pr_2(-1) = 0.5$

This then induces a probability distribution over cuts

A probabilistic cut: An ensemble of cuts

Probabilistic Cut: an assignment of binary random variables



Noncommutative Cut: an assignment of binary observables

Probabilistic Cut: an assignment of binary random variables

$Pr_1(+1) = 0.8$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $Pr_2(-1) = 0.5$

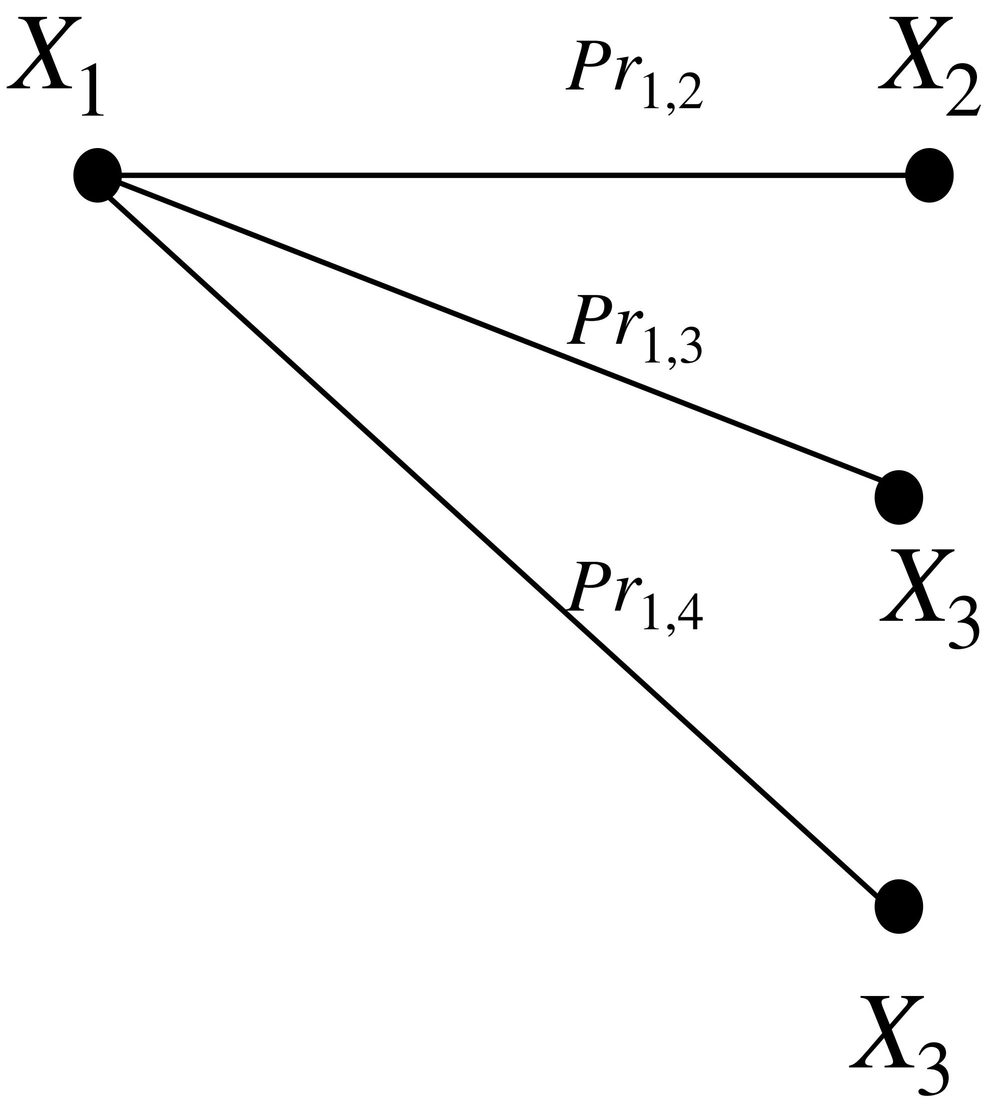
Noncommutative Cut: an assignment of binary observables

$$Pr_{12}(+1, +1) = 0.1$$
$$Pr_{12}(+1, -1) = 0.2$$
$$Pr_{12}(-1, +1) = 0.3$$
$$Pr_{12}(-1, -1) = 0.4$$

A probabilistic assignment induces a distribution on cuts

An observable assignment doesn't

Inconsistencies of Edge Probabilities



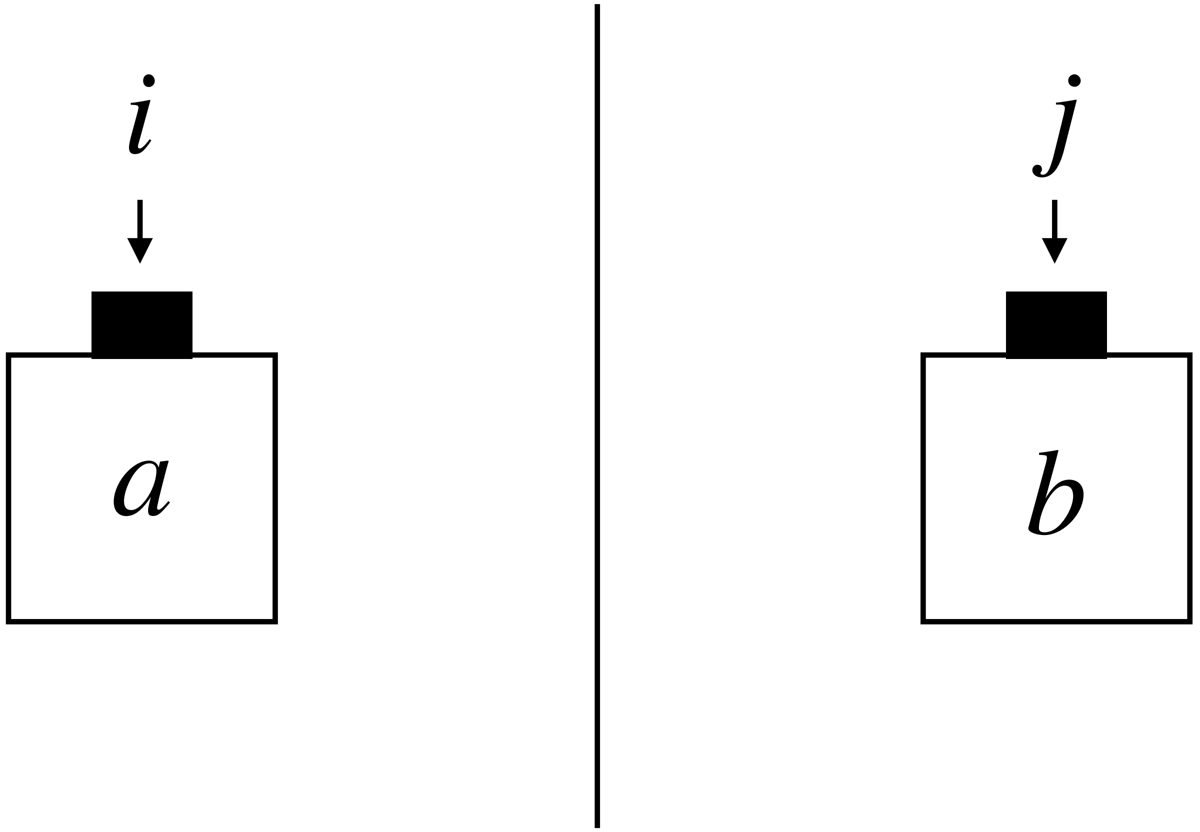
Warning: X and Y must commute for them to be simultaneously measurable

$$Pr_{12}(+1, +1) = 0.1$$
$$Pr_{12}(+1, -1) = 0.2$$
$$Pr_{12}(-1, +1) = 0.3$$
$$Pr_{12}(-1, -1) = 0.4$$

Warning: X and Y must commute for them to be simultaneously measurable

- Two formulations of noncommutative MaxCut:
  - If there is an edge the observables must commute: Quantum correlations: Q-MaxCut
  - Do not impose any commutation: The synchronous model of quantum correlations: NC-MaxCut

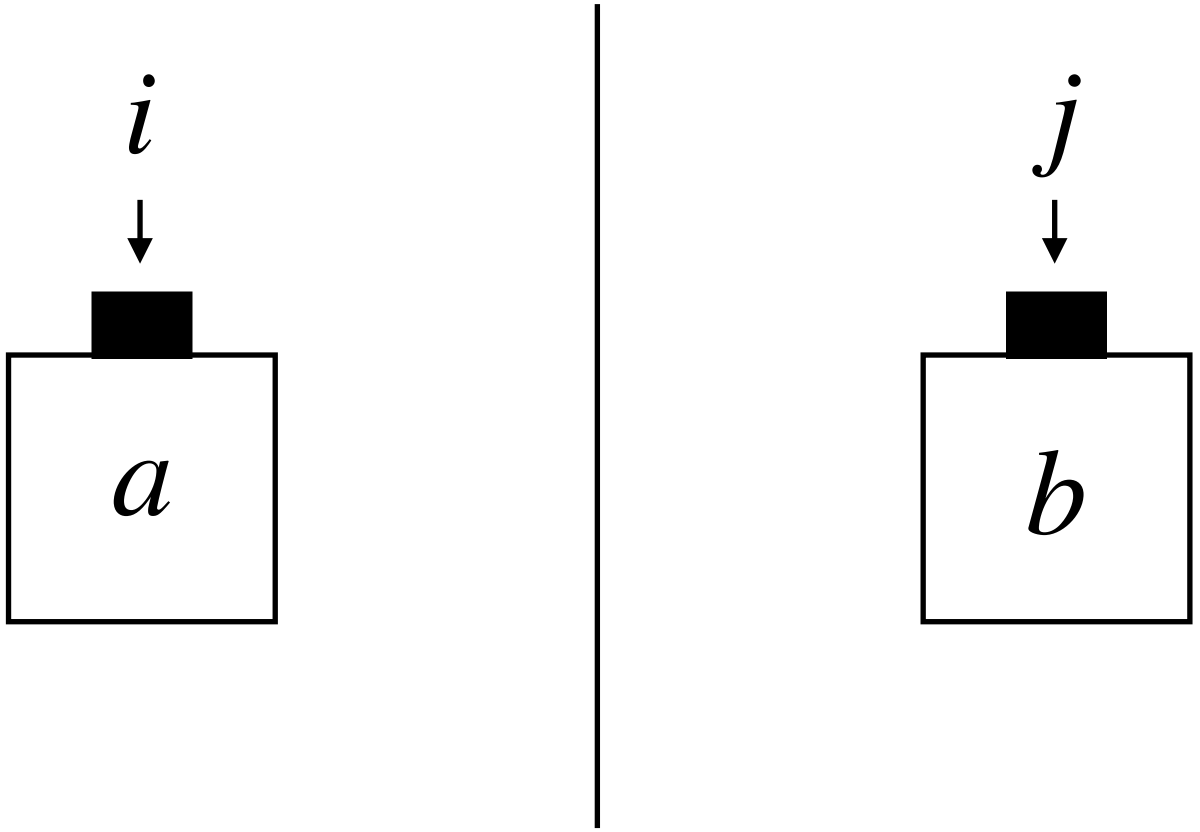
Operational Interpretation of Noncommutative Cuts



$$i, j \in V,$$

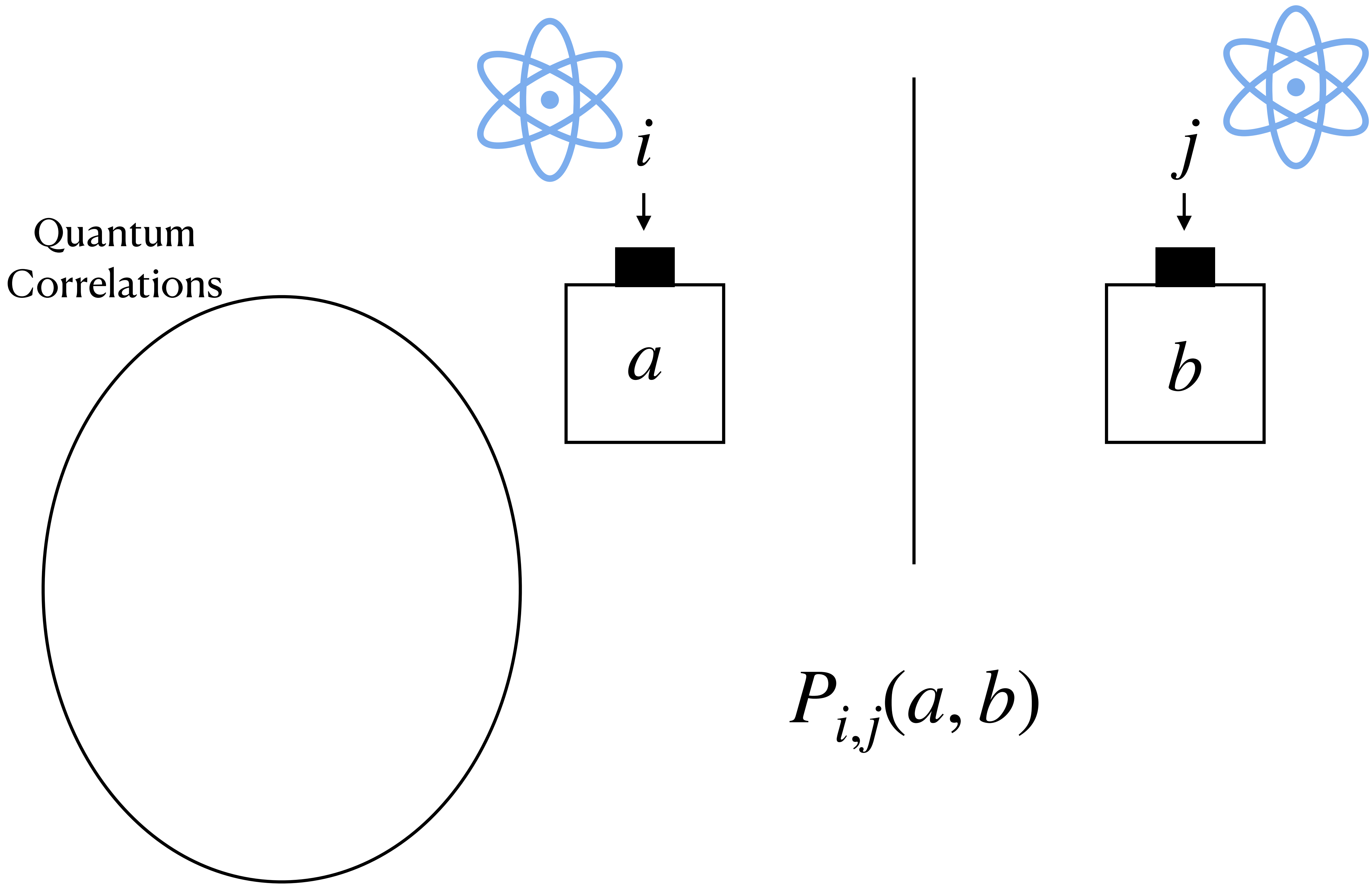
$$a, b \in \{+1, -1\}$$

Correlations



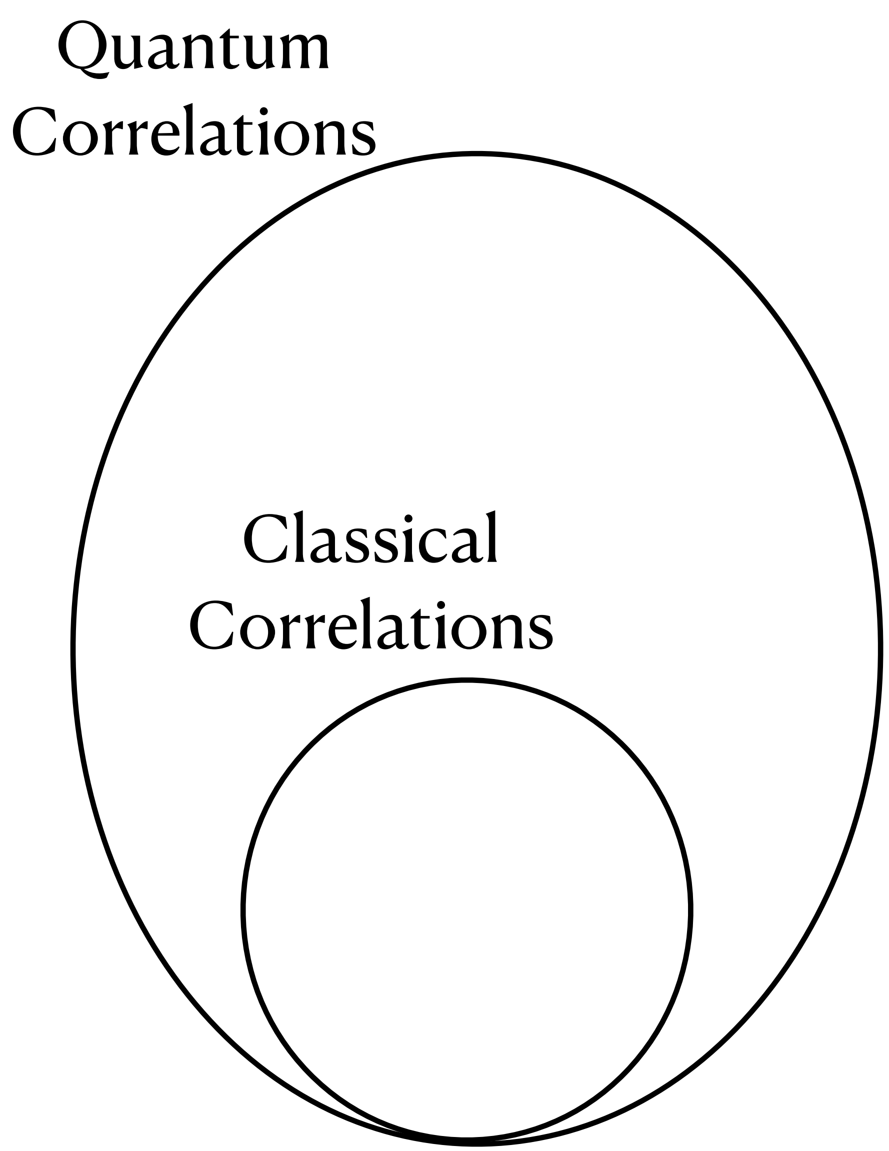
$$P_{i,j}(a, b)$$

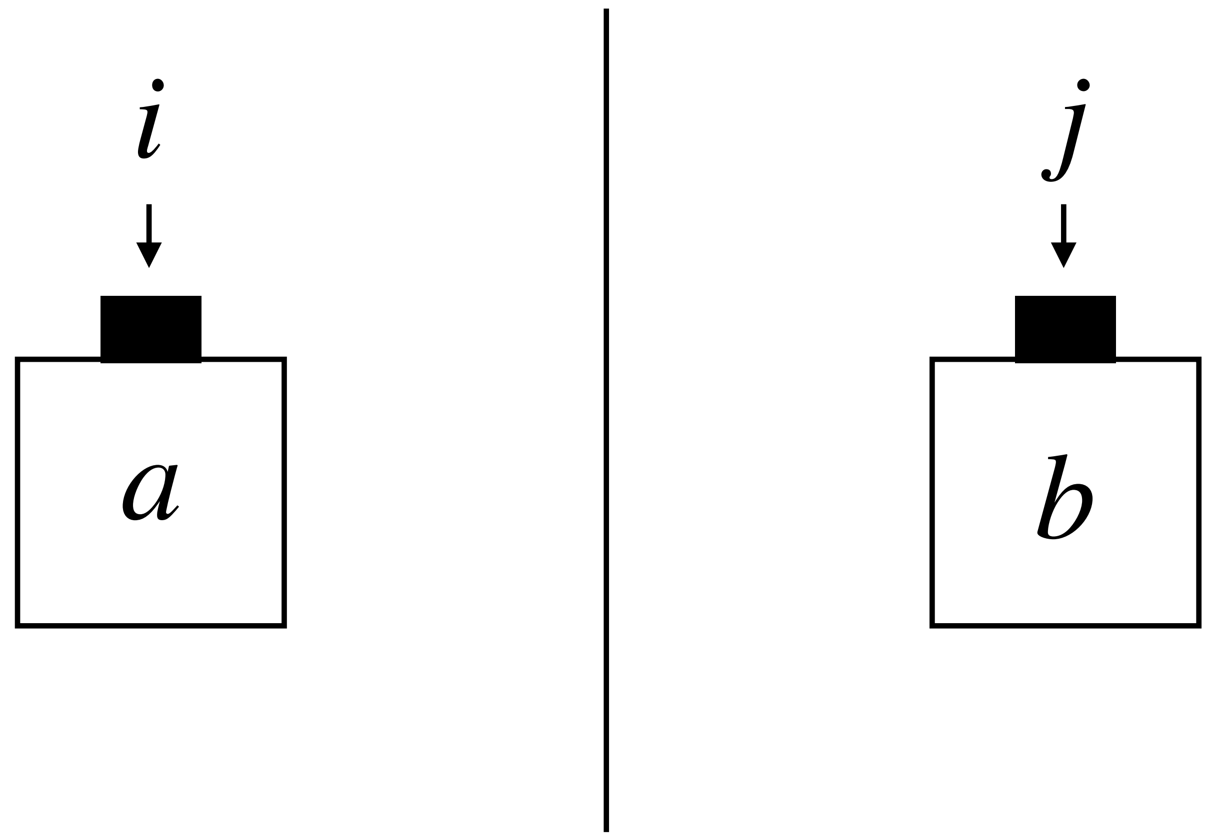
Quantum Correlations



$$P_{i,j}(a, b)$$

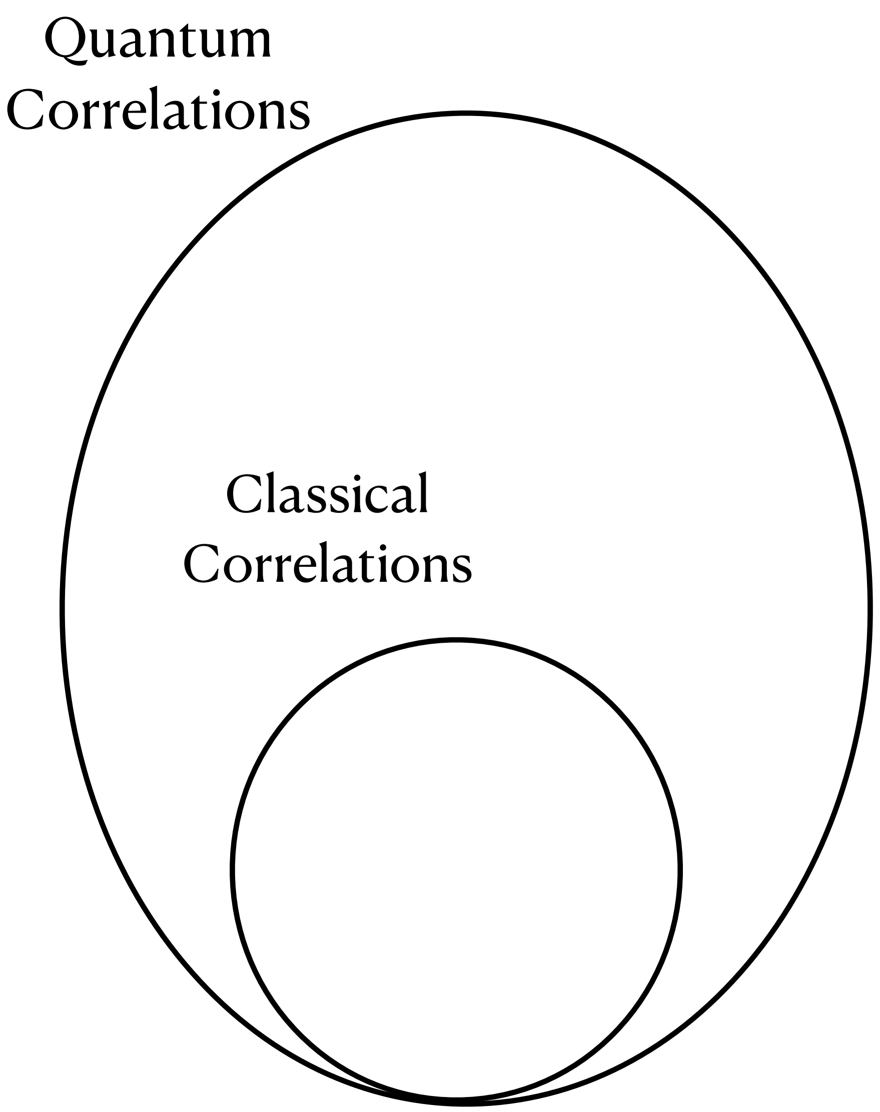
Classical Correlations



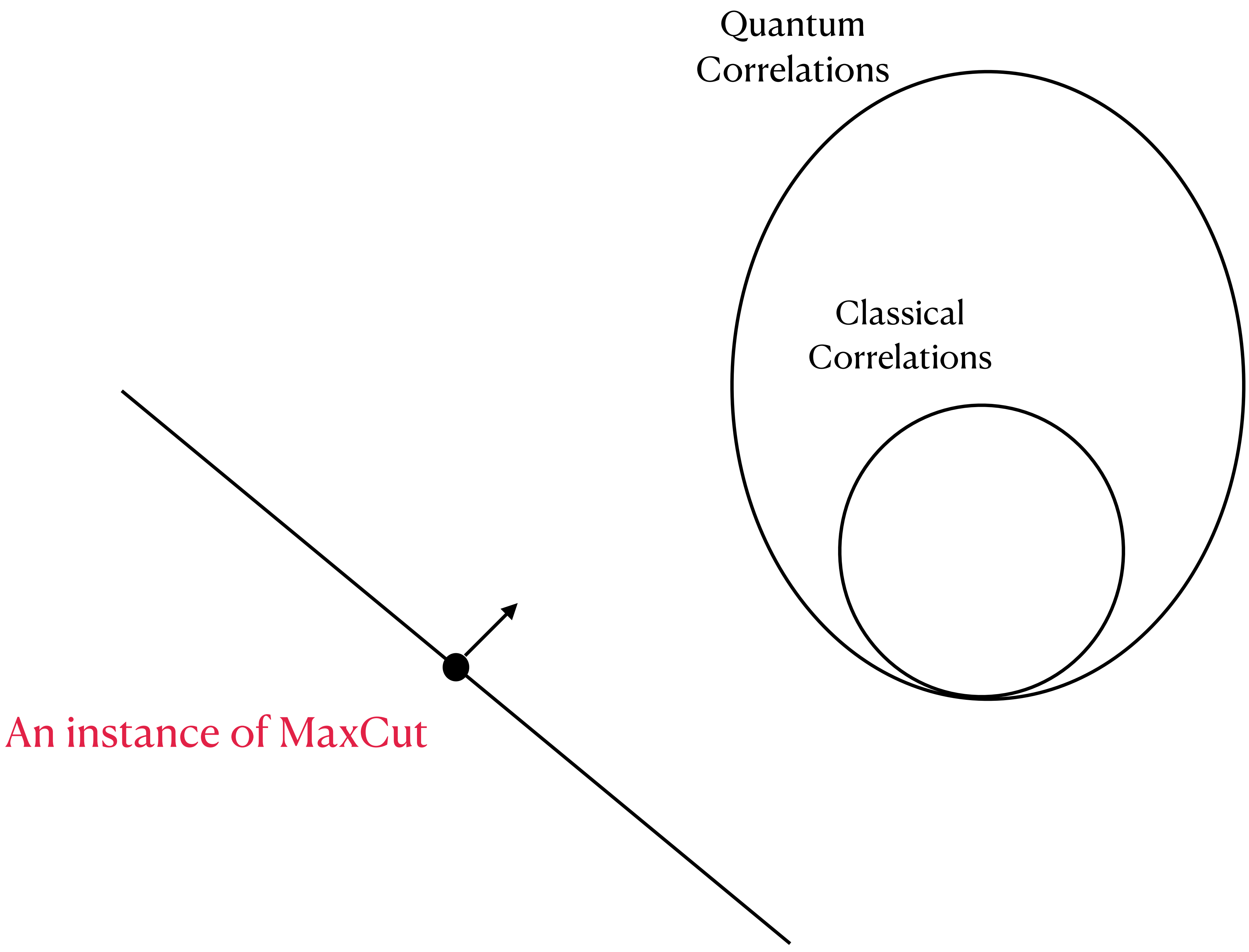


$$P_{i,j}(a, b)$$

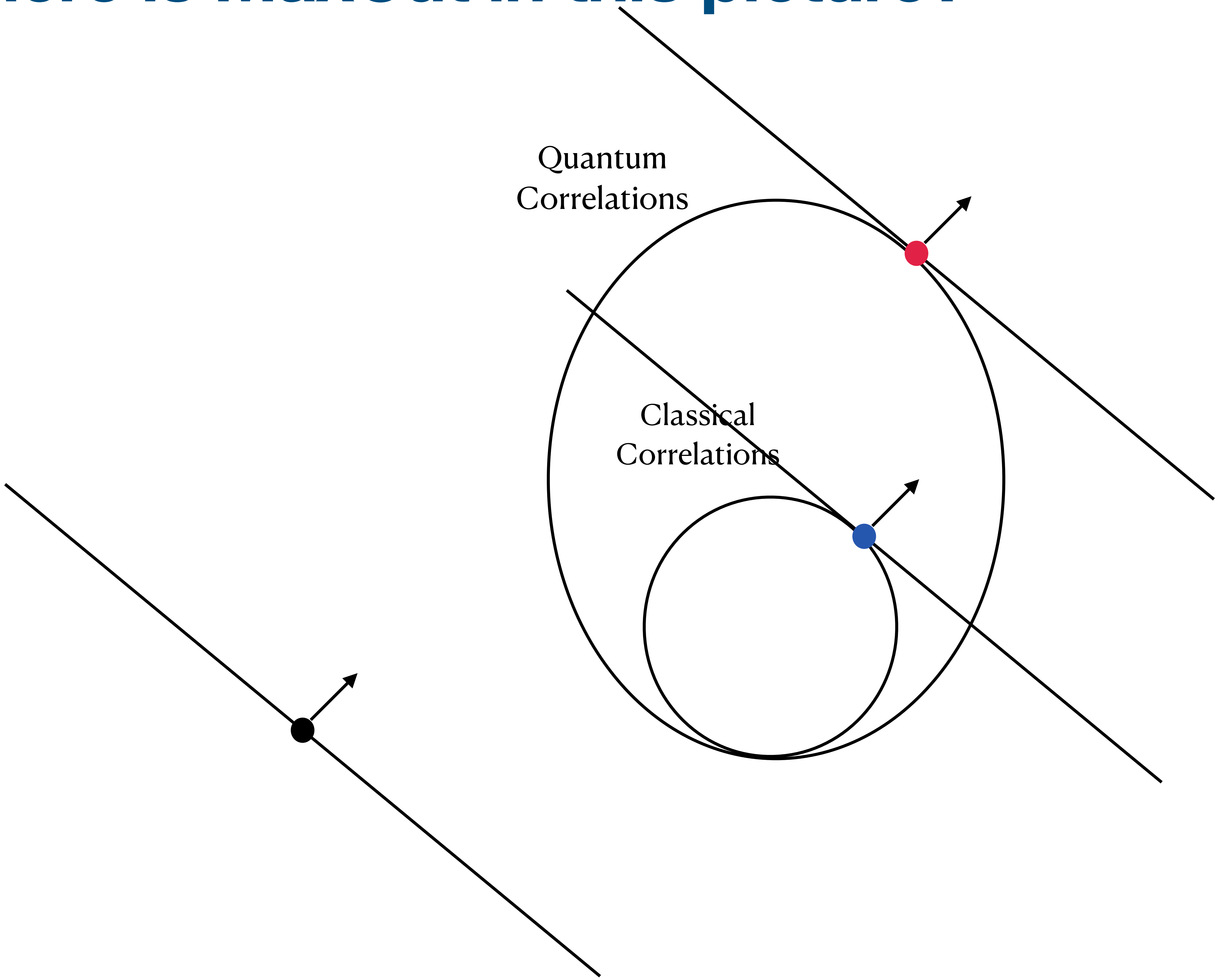
Where is MaxCut in this picture?



Where is MaxCut in this picture?



Where is MaxCut in this picture?



Computational Aspects

Computational Aspects

- Slofstra 2016: Membership problem for "Quantum Correlations" is undecidable
- In particular optimization over the set is uncomputable
- Ji, Natarajan, Vidick, Wright, Yuen 2020 (MIP*=RE): Even approximation is beyond reach
- So generic NC-CSPs are very complex

Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. $\quad X_i \quad$ is unitary with ± 1 eigenvalues

Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

- Karp 1972: MaxCut is NP-Complete

Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. $\quad X_i \quad$ is unitary with ± 1 eigenvalues

- Karp 1972: MaxCut is NP-Complete
- Tsirelson 1980: NC-MaxCut is in P

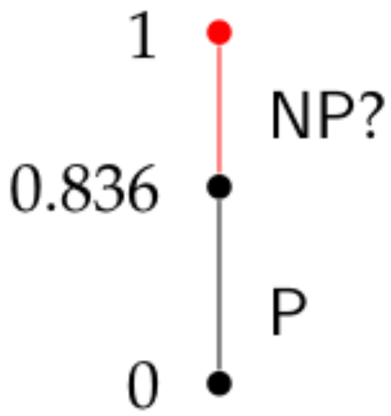
Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

$$\text{s.t.} \qquad X_i \quad \text{is unitary with } \pm 1 \text{ eigenvalues}$$

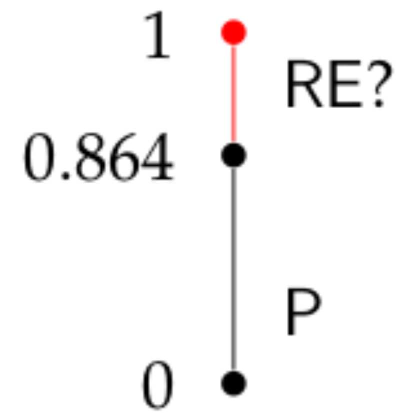
- Karp 1972: MaxCut is NP-Complete
- Tsirelson 1980: NC-MaxCut is in P
- The best classical algorithm is SDP rounding by Goemans and Williamson
- Tsirelson's algorithm is an operator generalization

What about other NC-CSPs?



(a) Max-3-Cut

Frieze and Jerrum



(b) Noncommutative Max-3-Cut

Culf, M., Spirig

Proof idea:

A strong property of noncommutative matrix algebras (rigidity or self-testing)

The power of noncommutative algebras in characterizing optimal solutions

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

The power of noncommutative algebras in characterizing optimal solutions

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

Rigidity: In every optimal solution X_{12} and X_{21} must anticommute

This is why in MaxCut and Max3Cut we do better in the NC setting

Hardness front

- PCP theorem: Approximating Label-Cover is NP-hard
  (Arora, Safra, Lund, Motwani, Sudan, Szegedy, Raz, Håstad)

- NC-PCP theorem (MIP*=RE): Approximating NC-Label-Cover is RE-hard
  (Ji, Natarajan, Vidick, Wright, Yuen 2020)

Hardness front

- PCP theorem: Approximating Label-Cover is NP-hard
  (Arora, Safra, Lund, Motwani, Sudan, Szegedy, Raz, Håstad)

- NC-PCP theorem (MIP*=RE): Approximating NC-Label-Cover is RE-hard
  (Ji, Natarajan, Vidick, Wright, Yuen 2020)

- Compare this with the situation for the Local Hamiltonian problem (LH):

  Quantum PCP conjecture: Approximating Local Hamiltonian is QMA-hard

Hardness front

- Similarly UGC has an NC-UGC analogue

- Assuming UGC, approximating MaxCut to better than 0.878 is NP-hard (Khot, Kindler, Mossel, O'Donnell)

- Assuming Q-UGC, approximating Q-MaxCut to better than 0.878 is RE-hard (M., Spirig)

A classical theorem involving NP and CSP

becomes

A theorem that involves RE and NC-CSP

What is the difference between

Local Hamiltonians

and

CSPs or NC-CSPs ?

The most general notion of a CSP

- The notion of locale, site, or variable
    - boolean ± 1 in MaxCut
    - binary observable in NC-MaxCut
    - qubit in Local Hamiltonians

The most general notion of a CSP

- The notion of locale, site, or variable
  - boolean ± 1 in MaxCut
  - binary observable in NC-MaxCut
  - qubit in Local Hamiltonians
- Constraints
  - operations on variables
    - MaxCut: addition and multiplication (commutative)
    - NC-MaxCut: addition and multiplication (noncommutative)

The most general notion of a CSP

- The notion of locale, site, or variable
  - boolean ± 1 in MaxCut
  - binary observable in NC-MaxCut
  - qubit in Local Hamiltonians
- Constraints
  - operations on variables
    - MaxCut: addition and multiplication (commutative)
    - NC-MaxCut: addition and multiplication (noncommutative)
  - interactions between sites
    - Local Hamiltonians: the Hamiltonian

CSPs: commutative algebras

NC-CSPs: matrix algebras

Local Hamiltonians: not algebraic

The algebraic nature of CS tools (sum-check protocol, low-degree testing, Fourier analysis on the hypercube)

fits

the algebraic nature of CSPs and NC-CSPs

A proposal

NC-CSPs are expressive

P NP RE

2SAT 3SAT (Cook levin) NC--3SAT (Ji, Natarajan, Vidick, Wright, Yuen)

NC-CSPs are expressive

P — NP — RE — Π_2

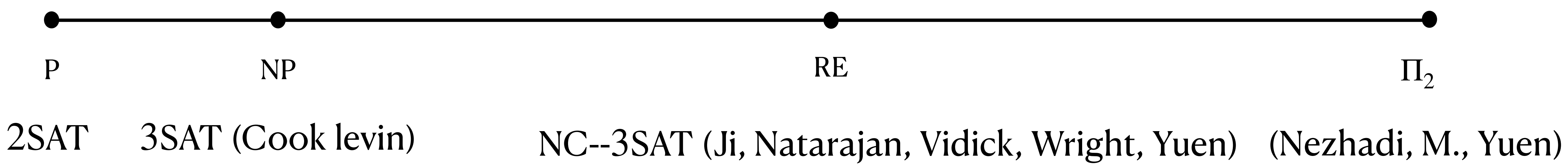
2SAT

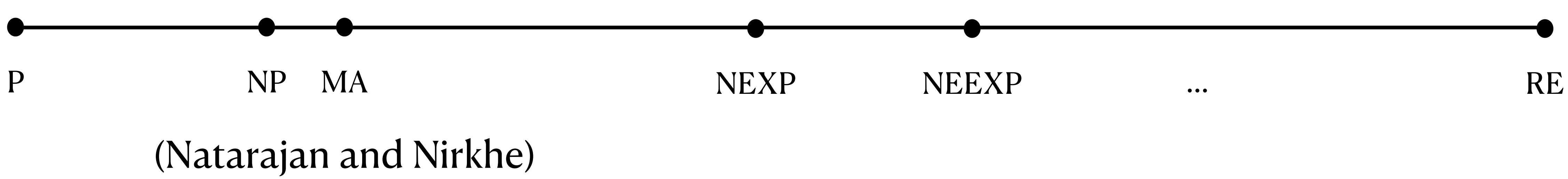
3SAT (Cook levin)

NC--3SAT (Ji, Natarajan, Vidick, Wright, Yuen)

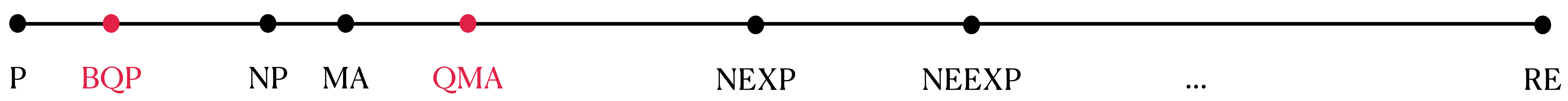
(Nezhadi, M., Yuen)

NC-CSPs are expressive

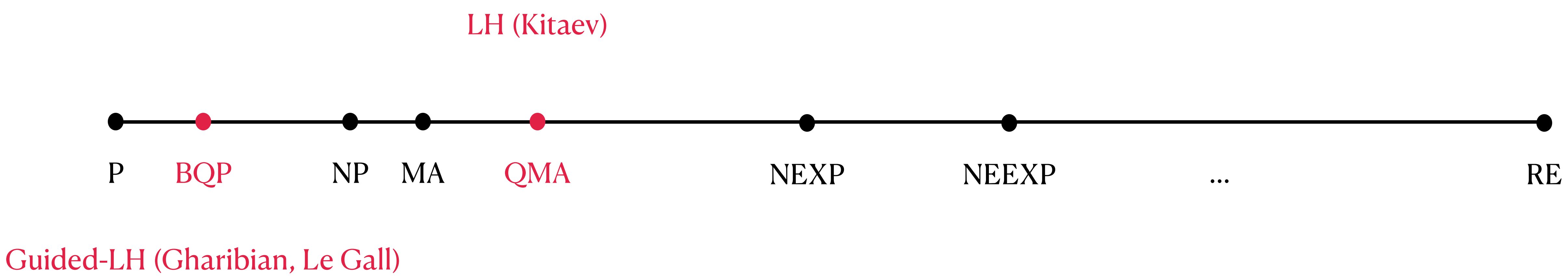




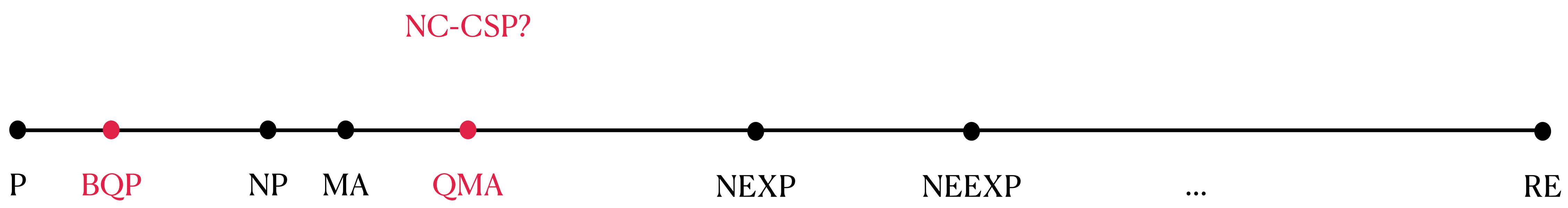
But they skip on quantum complexity classes



Local-Hamiltonian fills the gap

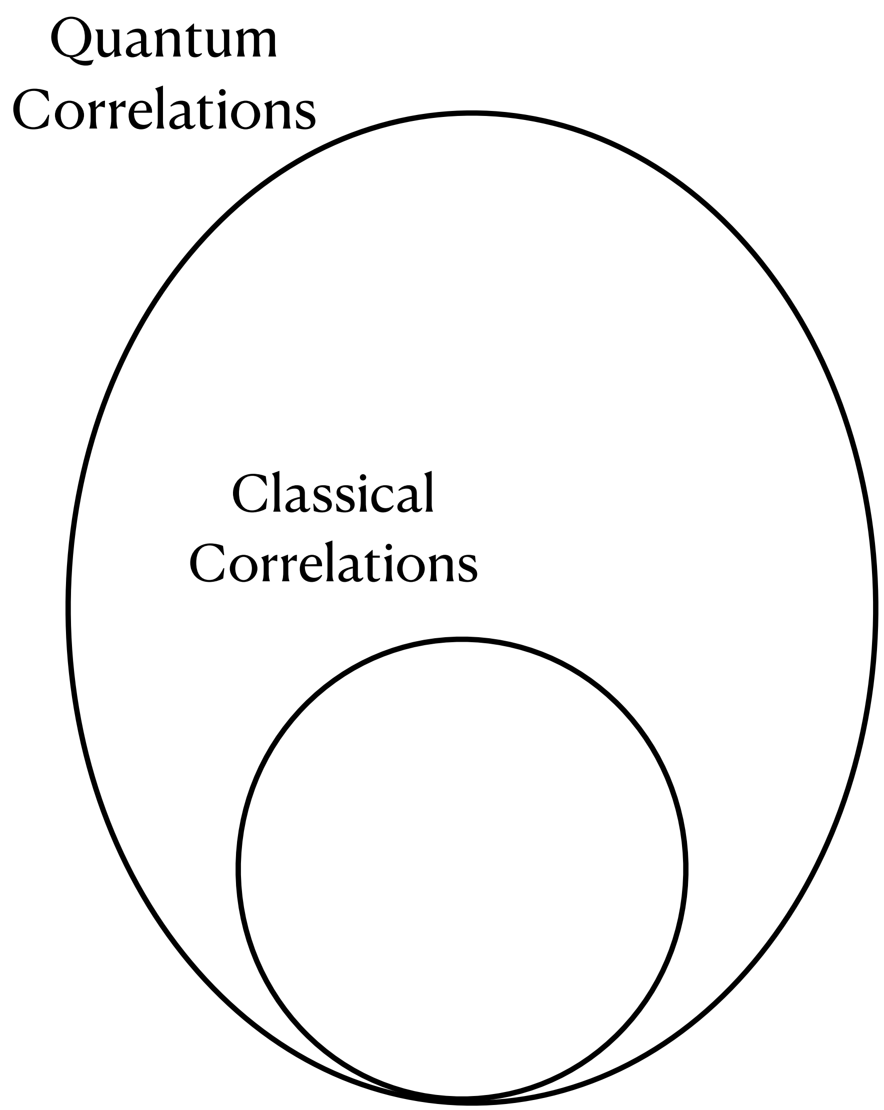


Open problem

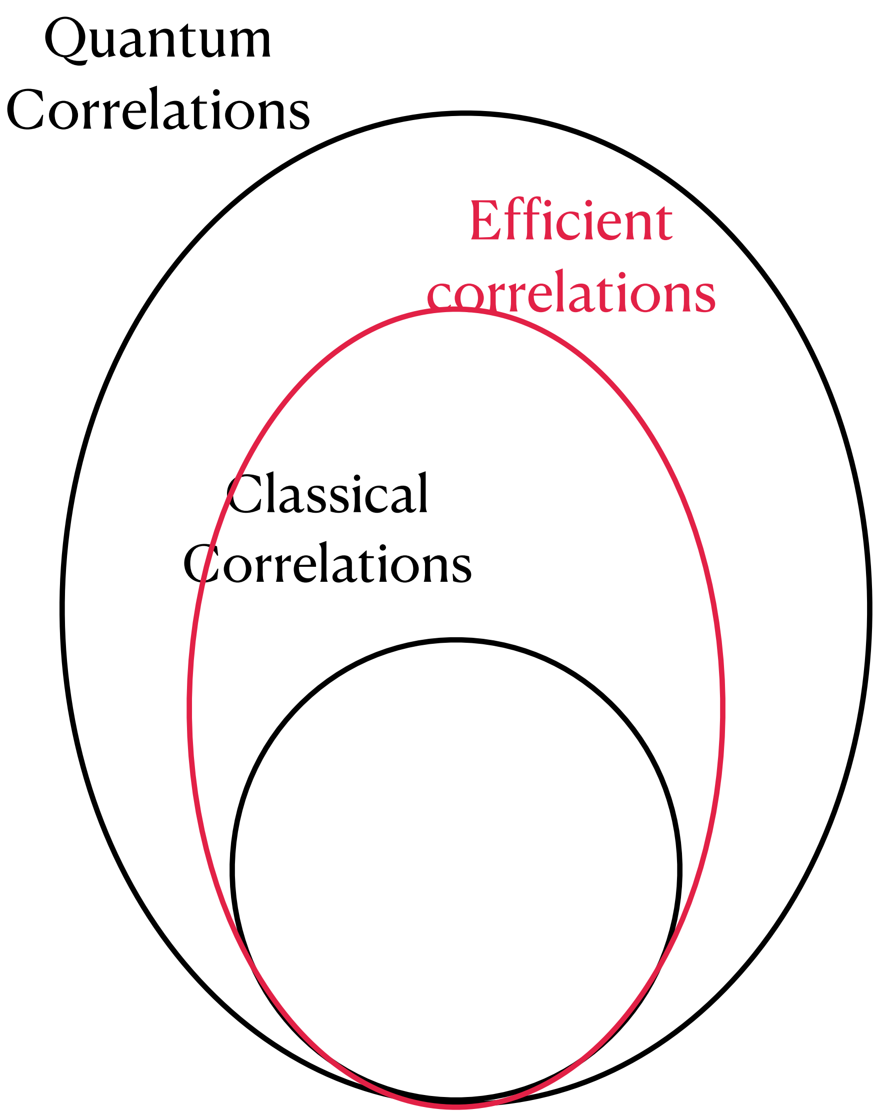


- Restricting the dimension of observable => nondeterministic classes
- Requiring that the observables are efficiently implementable (in BQP)

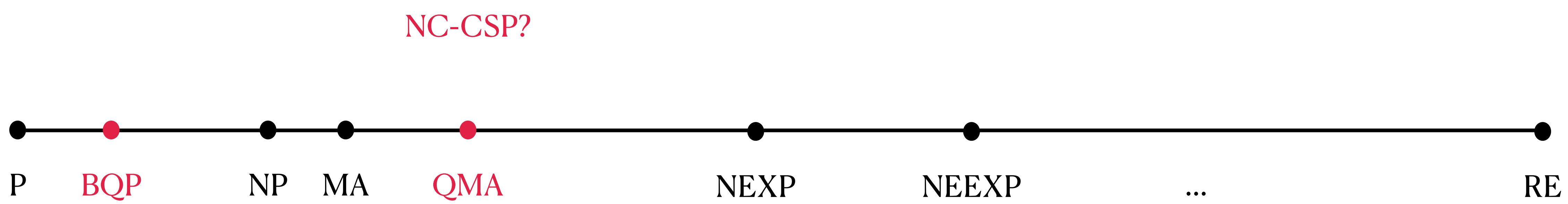
Remember this picture?



Remember this picture?



Open problem



$$\max \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

and X_i has an efficient circuit

- Two generalization of CSPs in quantum information
    - Local Hamiltonians
    - NC-CSPs
- NC-CSPs share the algebraicity of classical CSPs
- We have been able to reach almost the same maturity in NC-CSPs
- Many of the CS tools applicable to CSPs are algebraic in nature
- For Local Hamiltonian we need to invent new tools
- But QMA we may be able to understand better
    - if we find an NC-CSP that captures it!