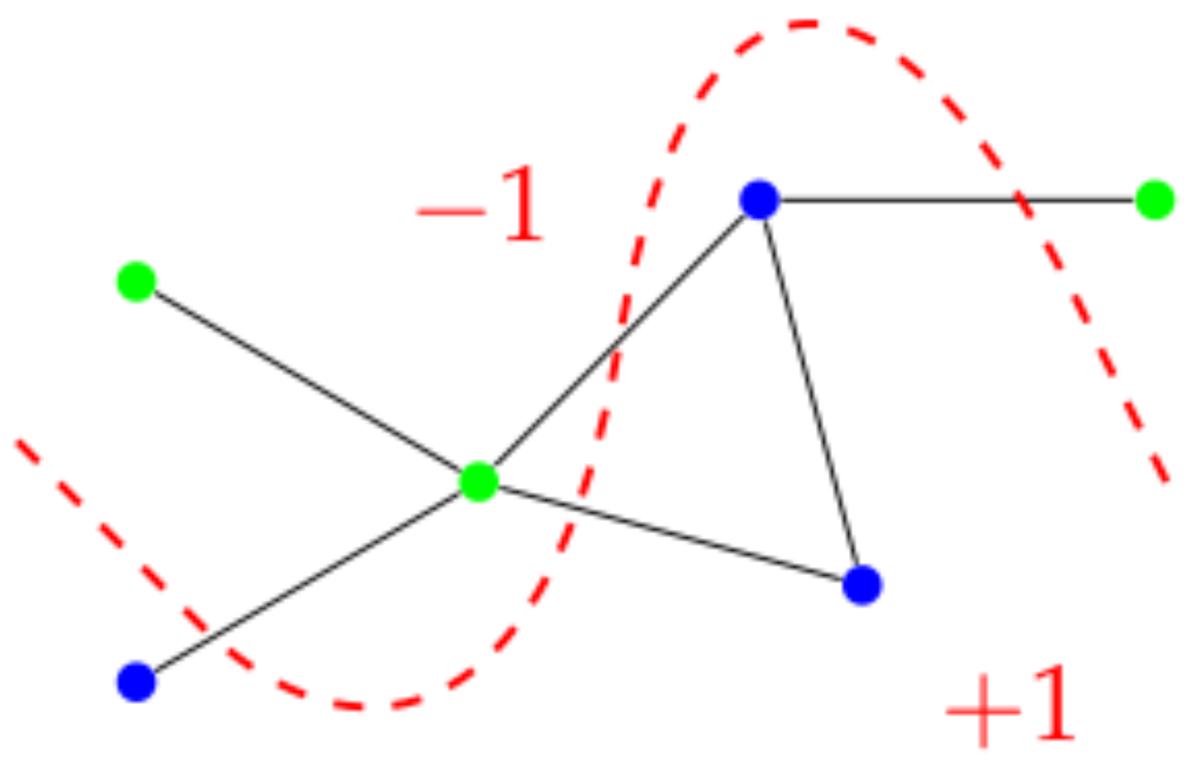
Constraint satisfaction in the quantum setting

Hamoon Mousavi, Stony Brook 1/24/25

MaxCut



$$\max \quad \sum_{(i,j)\in E} \gamma_{ij} x_i x_j$$

$$\text{s.t.} \quad x_i \text{ are } -1 \text{ or } +1$$

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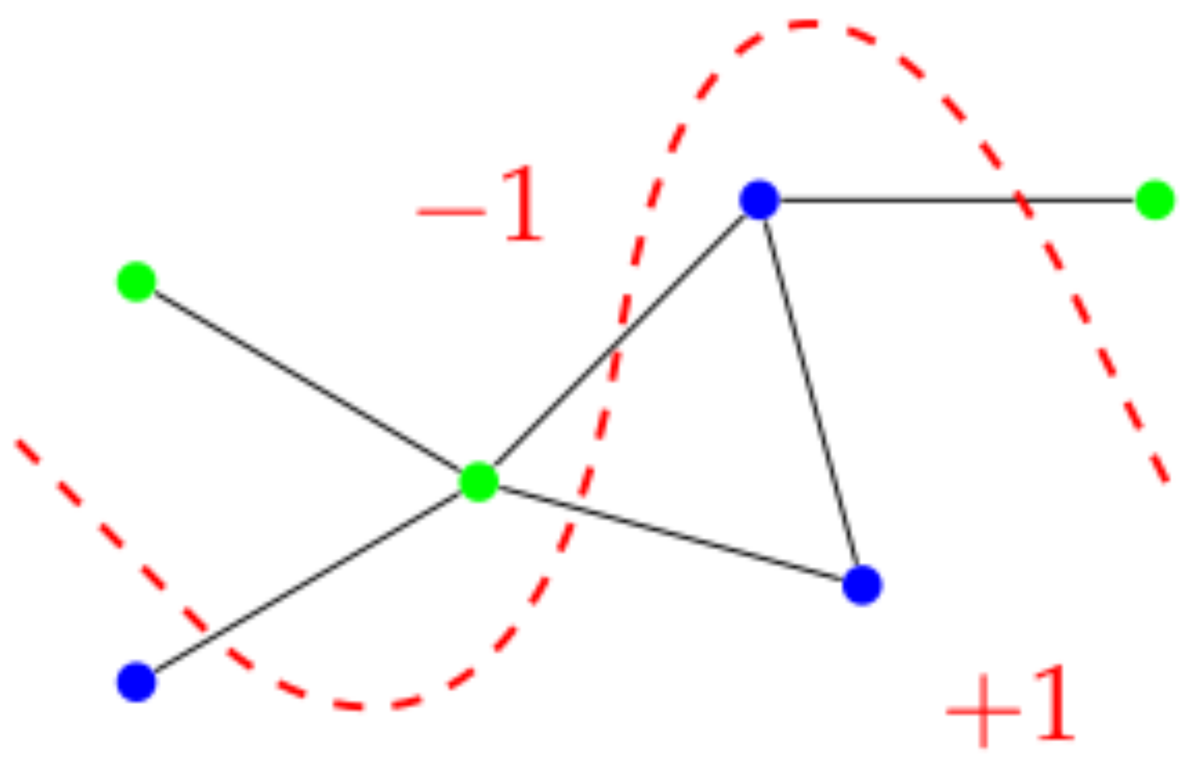


$$\max \sum_{(i,j)\in E} \gamma_{ij} x_i x_j$$

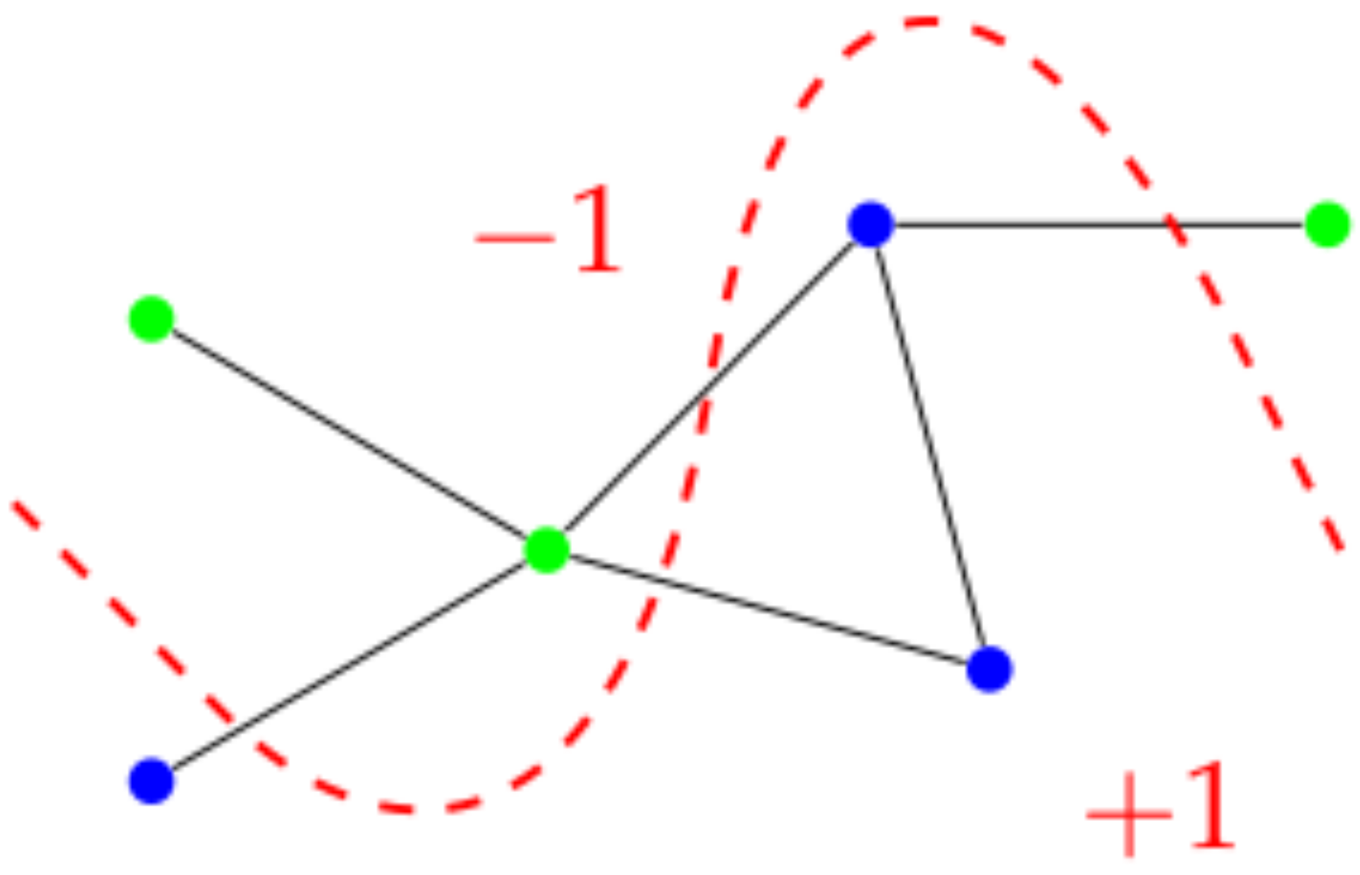
s.t. x_i are -1 or $+1$

Noncommutative or operator or quantum MaxCut

$$\max \operatorname{tr}\left(\sum_{(i,j)\in E} \gamma_{ij} X_i X_j \right)$$

s.t. eigenvalues of X_i are -1 or $+1$

Natalie: When we hear the word 'cut' in a graph, we all imagine a picture, do we not?



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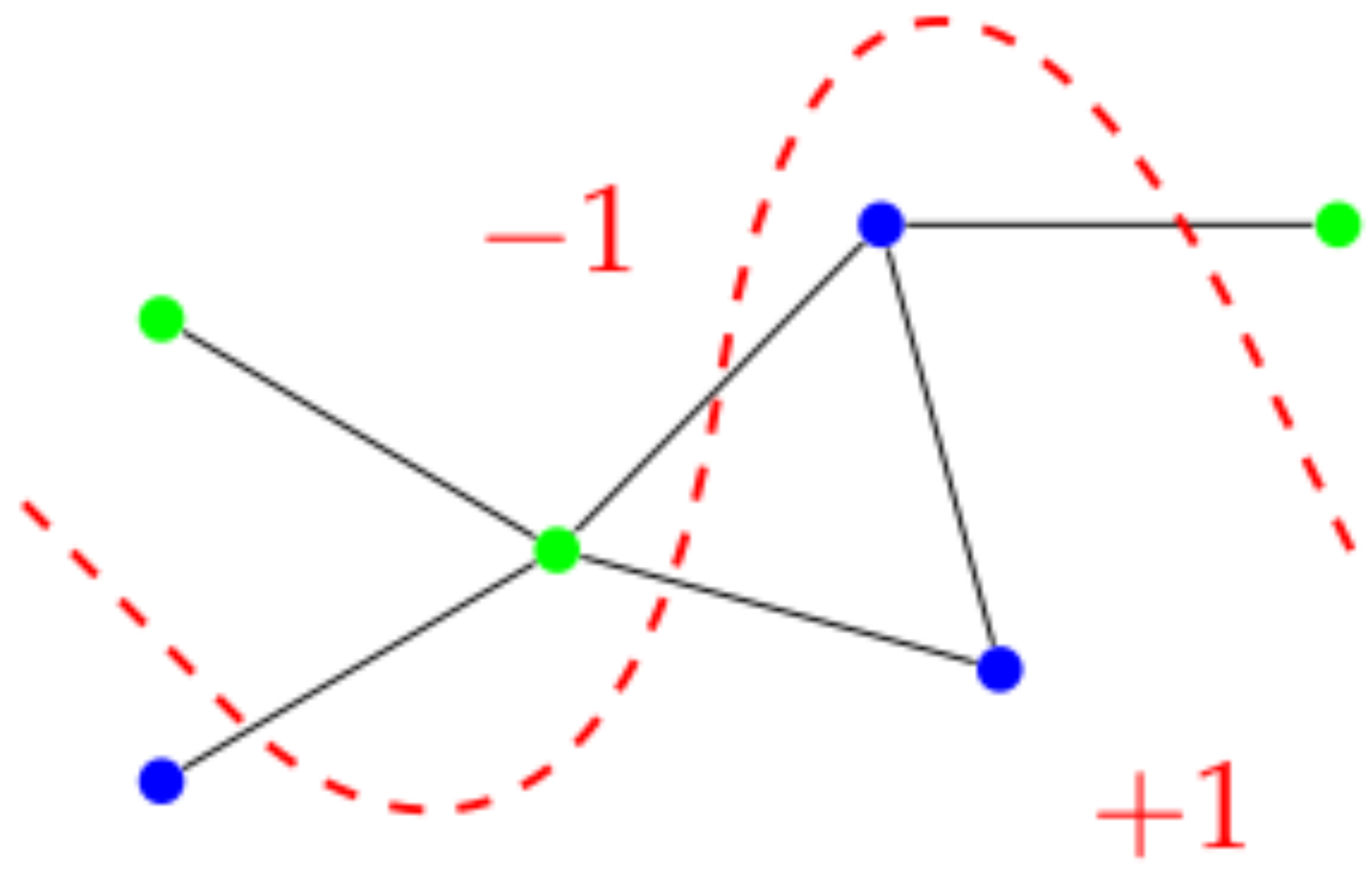

Then what form has 'quantum cut' got?

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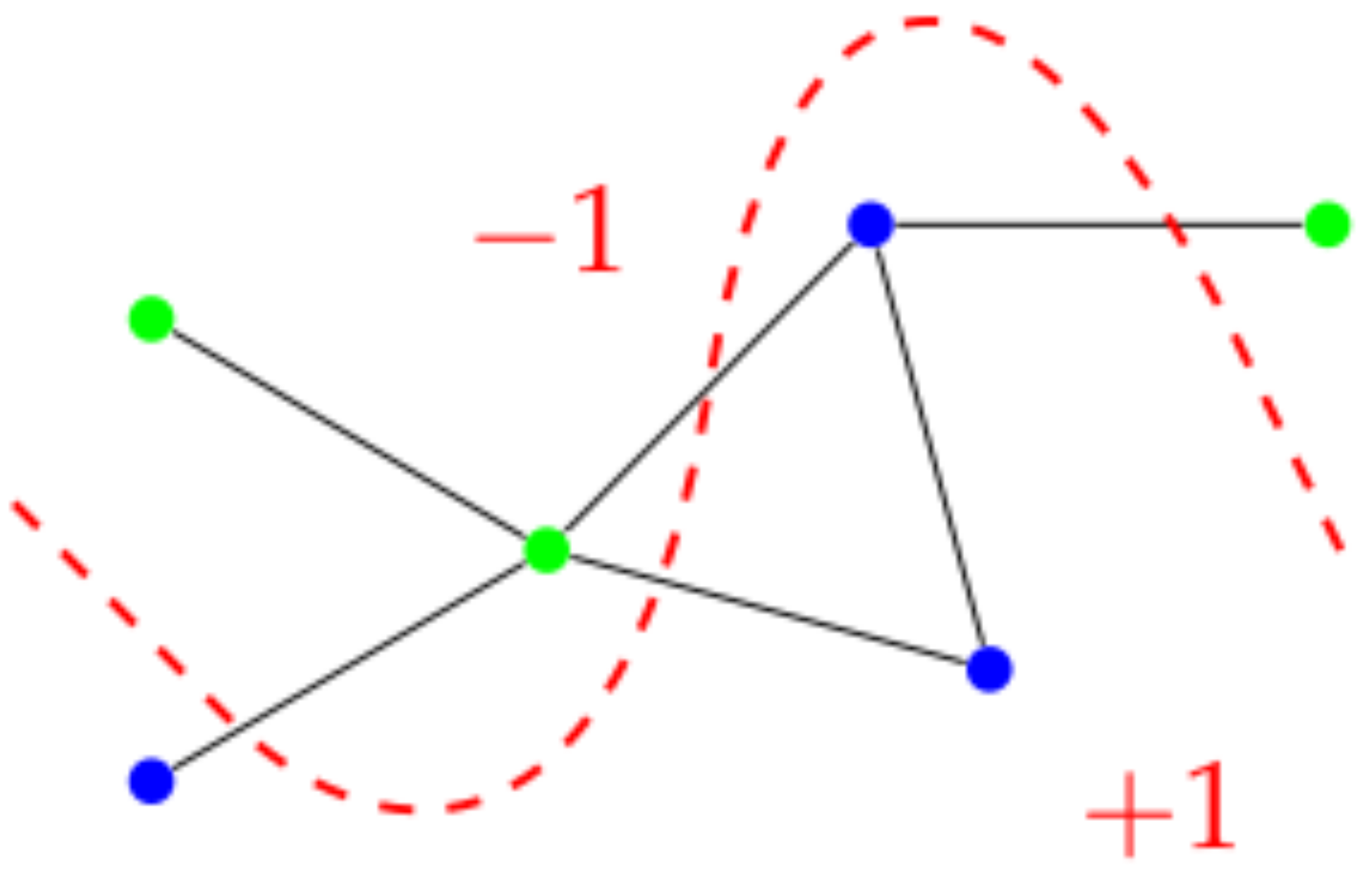

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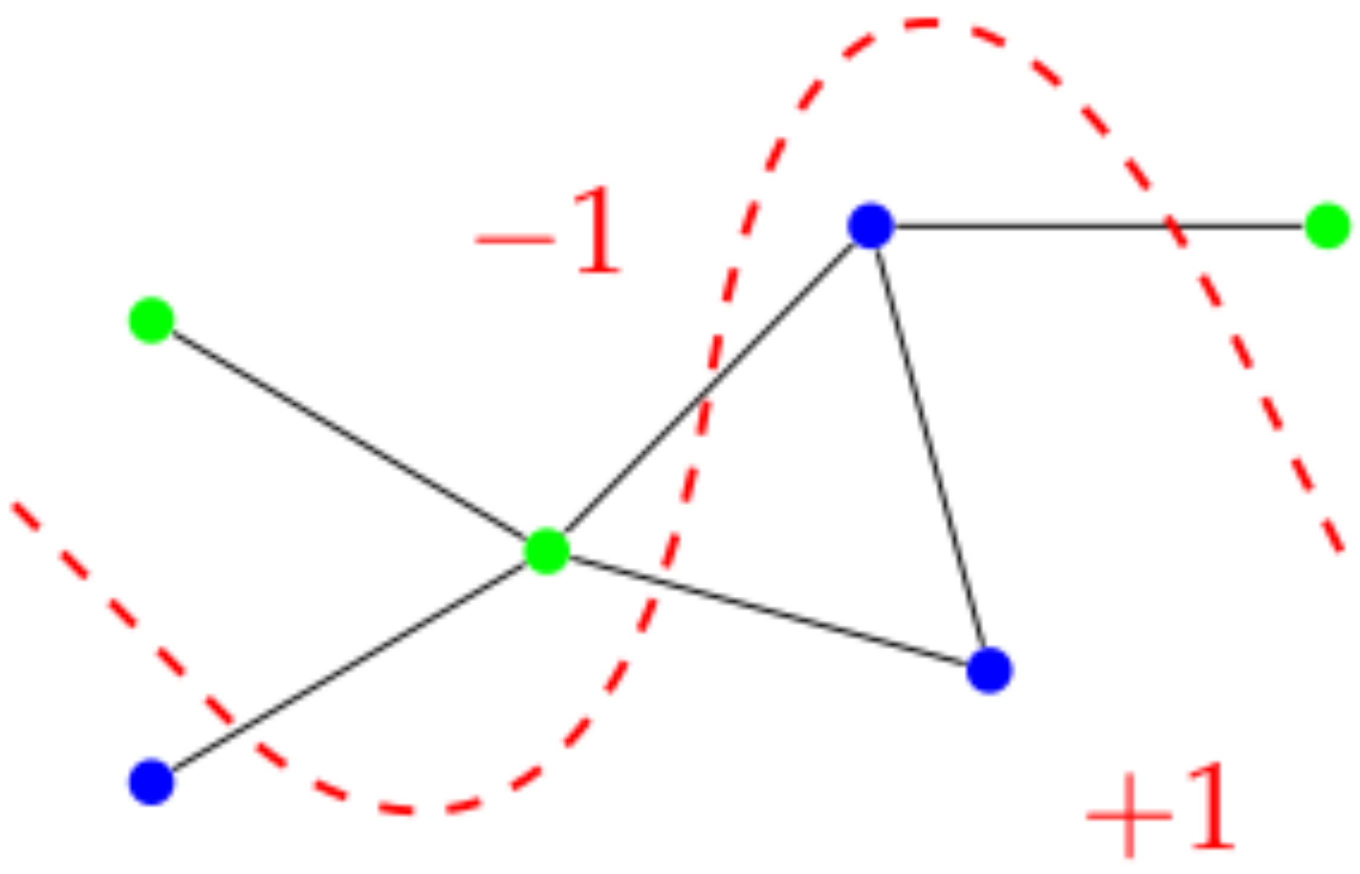

maximize:

subject to:

Timeline

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- 2024-2025 (new directions): Use these operator optimization problems to better understand
  - **Classical objects** (*unique games conjecture and plurality-is-stablest conjecture*)
  - **Quantum complexity classes** (*quantum NP*)

Plan

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  A. Quick review of classical constraint satisfaction problems (CSPs)

  B. Core message of the talk

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  What are the algorithmic results? What is their theory of hardness of approximation? And how is that different from the classical theory?

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- Open Problems: What is remained to do for OP-CSPs?

- Where to take this next: OP-CSPs can be useful in probing complexity classes we actually care about.

Intro

Quick intro to constraint satisfaction (classical)

Core message

Constraint Satisfaction Problems (CSPs)

Variables $x_1, x_2, \ldots, x_n$ taking values in a finite alphabet

and a number of constraints imposed on them, e.g. $x_1 - x_2 = 1$.

We think of them as optimization problems: Find an assignment that satisfies the most number of constraints.

When we say we can approximate CSP X to an approximation ratio of $\alpha \in (0,1)$, it means that there is a polynomial-time algorithm that is guaranteed to find an assignment satisfying $\alpha \cdot \mathrm{OPT}$ of the constraints.

Here OPT is the maximum number of constraints that can be satisfied.

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Classical Constraint Satisfaction

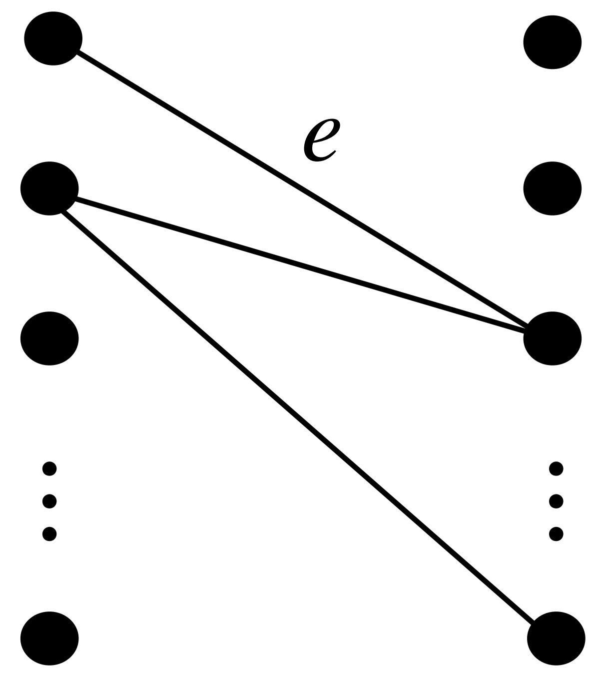
Examples: 3SAT, LabelCover , LinearSystems, ...

Classical Constraint Satisfaction

LabelCover

Labels

1,2,3,4



$\pi_e : \{1,2,3,4\} \to \{1,2,3,4\}$

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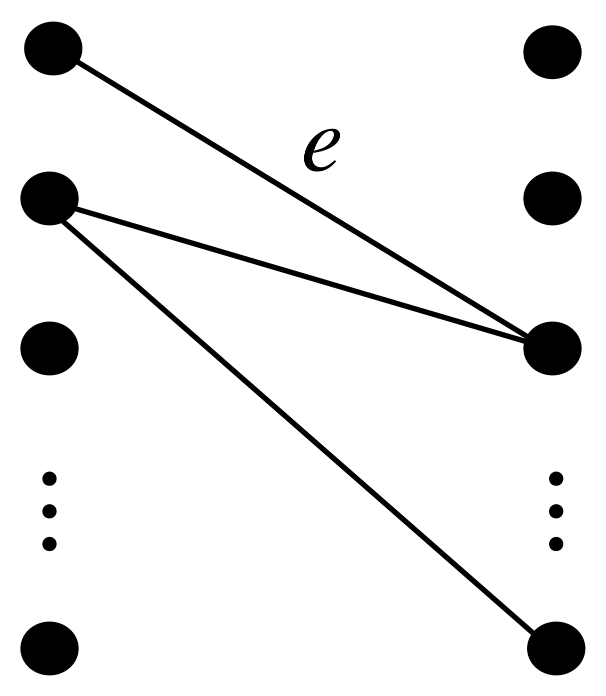

$\pi_e : \{1,2,3,4\} \rightarrow \{1,2,3,4\}$

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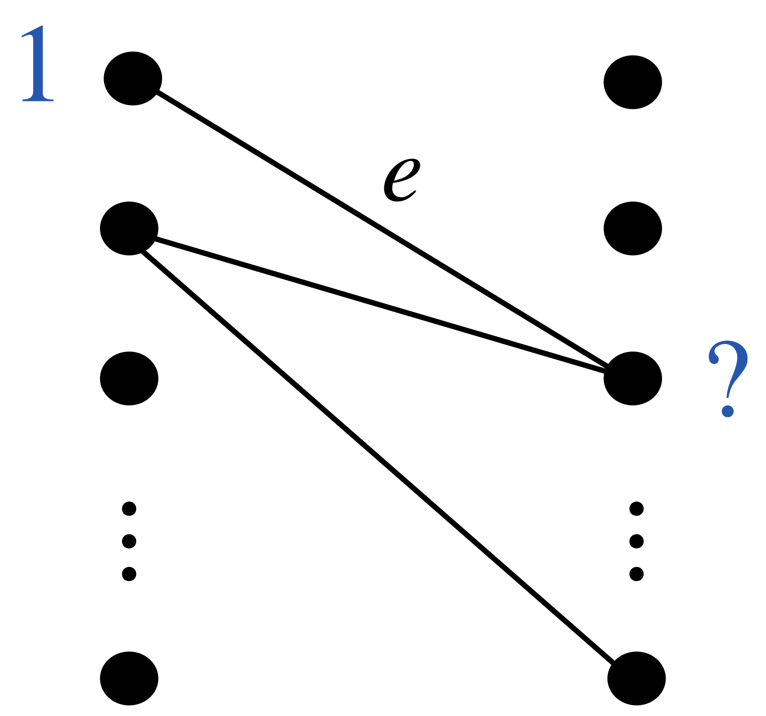

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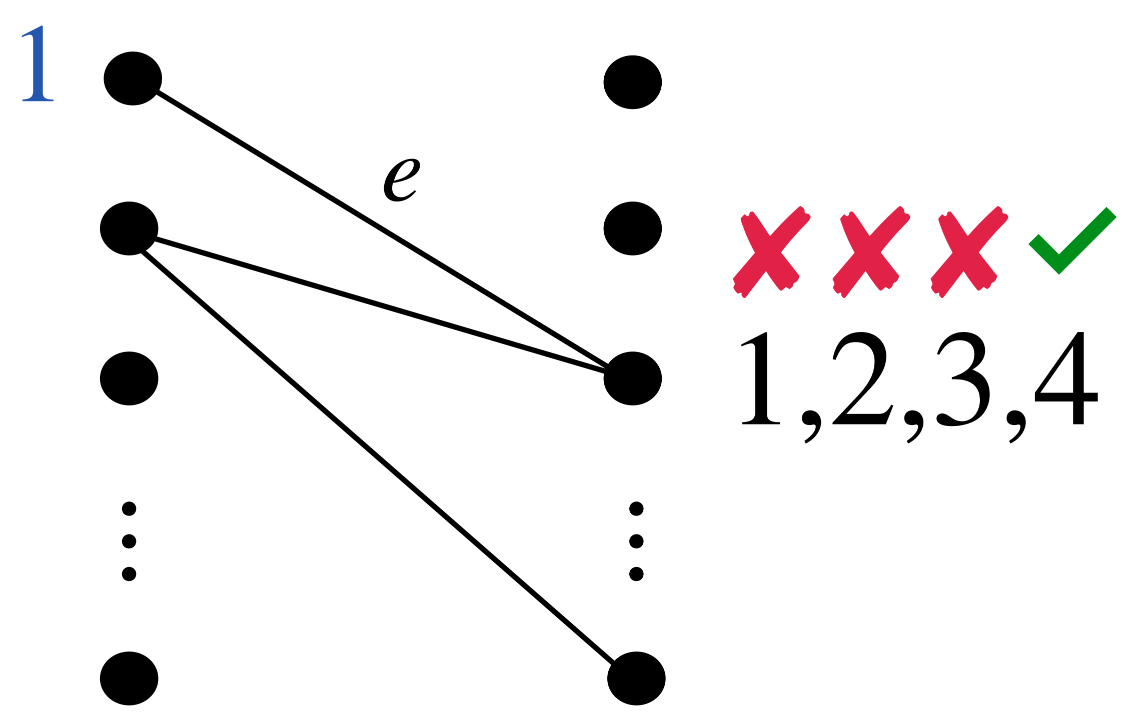

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UniqueLabelCover

special case where all π_e are one-to-one

The **PCP** theorem is a statement about the **LabelCover** problem:

It is NP-hard to approximate **LabelCover** to any constant approximation ratio.

The **unique games conjecture (UGC)** is a statement about the **UniqueLabelCover** problem:

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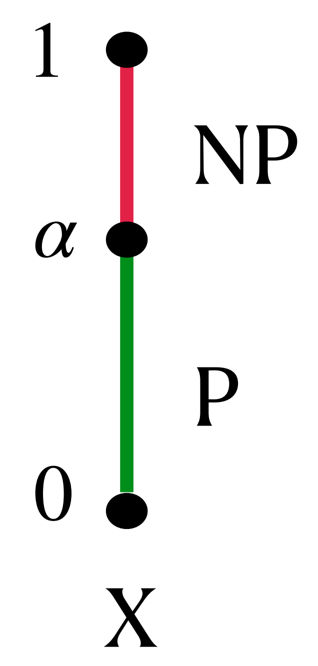
Quick intro to constraint satisfaction (classical)

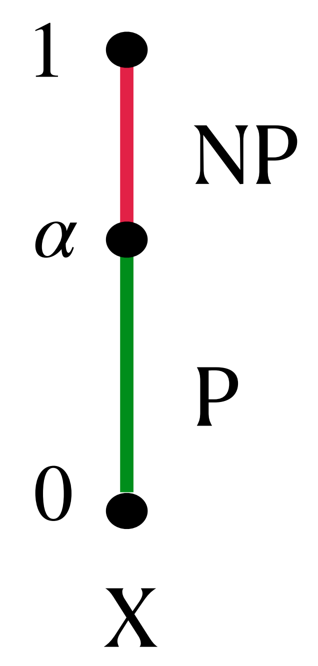
Core message

- CSP X is polynomial-time to approximate upto an approximation ratio of α. Approximating X beyond α is NP-hard.



- CSP X is polynomial-time to approximate upto an approximation ratio of α.

  Approximating X beyond α is NP-hard.
  Sometimes the hardness result is an implication of the **PCP theorem**, and some other times an implication of the **unique games conjecture**.



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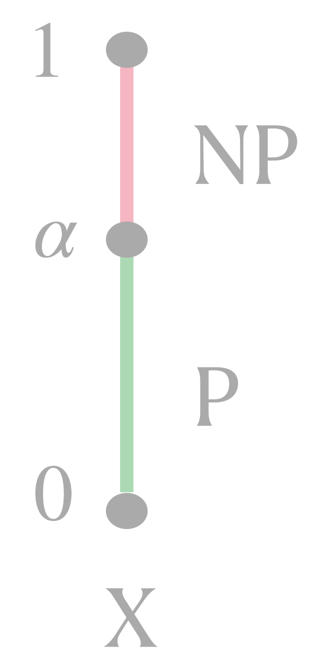

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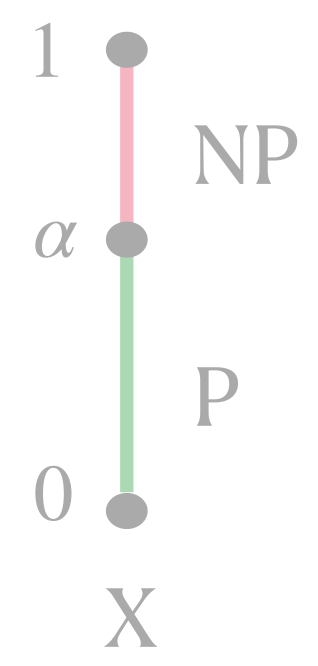
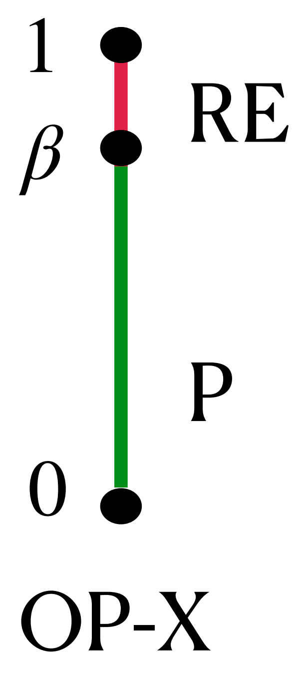


- There is a higher-dimensional operator extension of X called *operator-X* (or OP-X for short) of importance in quantum information.

  OP-X is polynomial-time to approximate upto an approximation ratio of β.
  Approximating X beyond β is undecidable (RE-hard).



(RE stands for recursively enumerable. The Halting Problem is complete for the class RE.)

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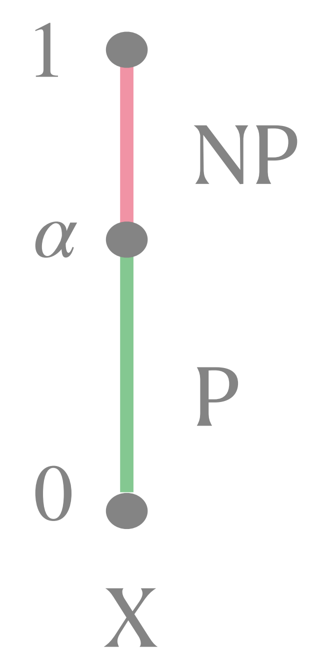

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  OP-X is polynomial-time to approximate upto an approximation ratio of β.
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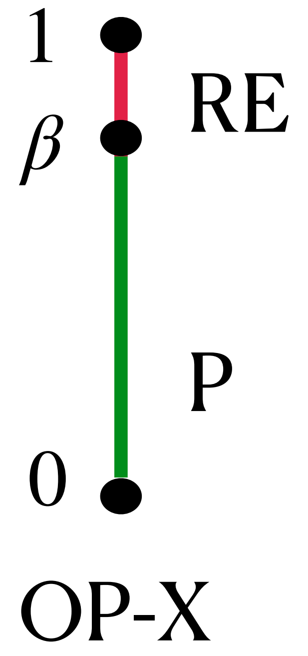
  Sometimes the hardness result is an implication of the **operator PCP theorem**, and some other times an implication of the **operator unique games conjecture**.



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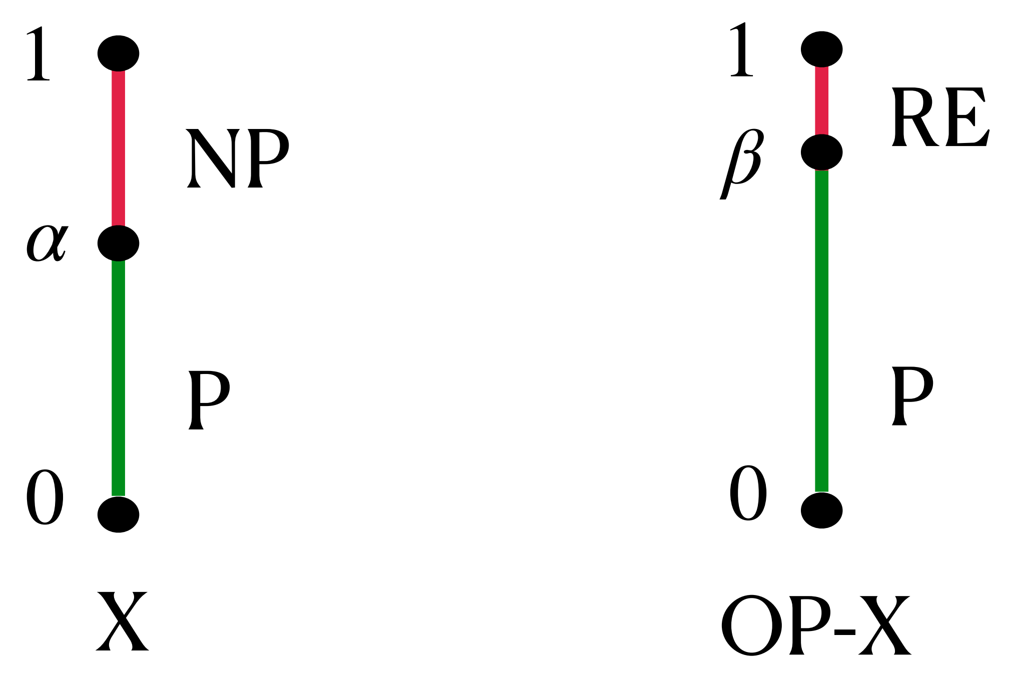
Complexity Landscape



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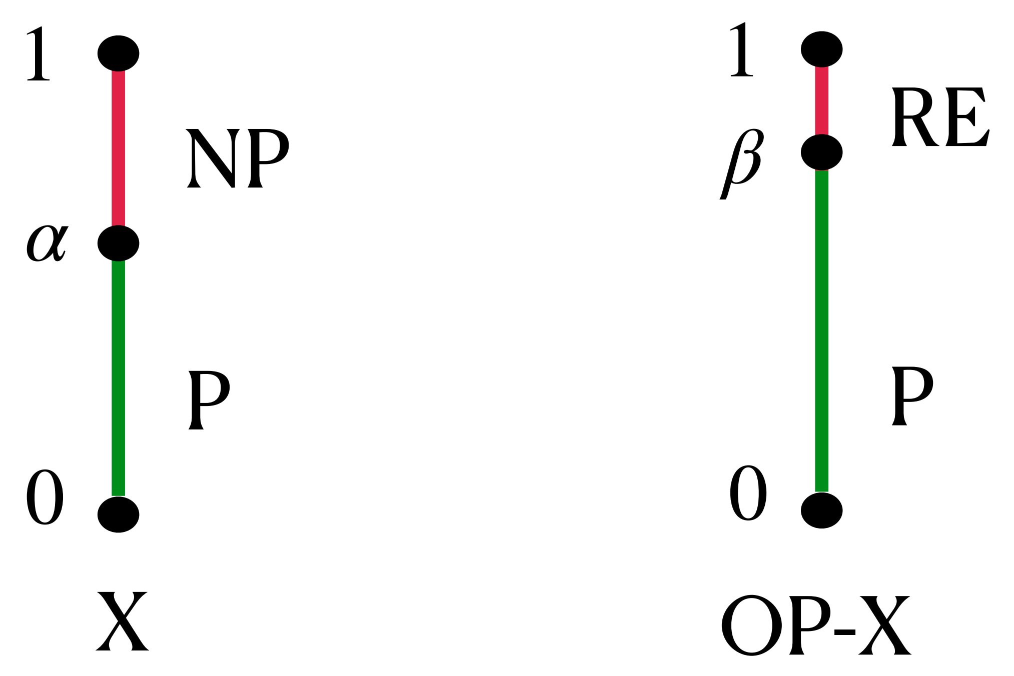
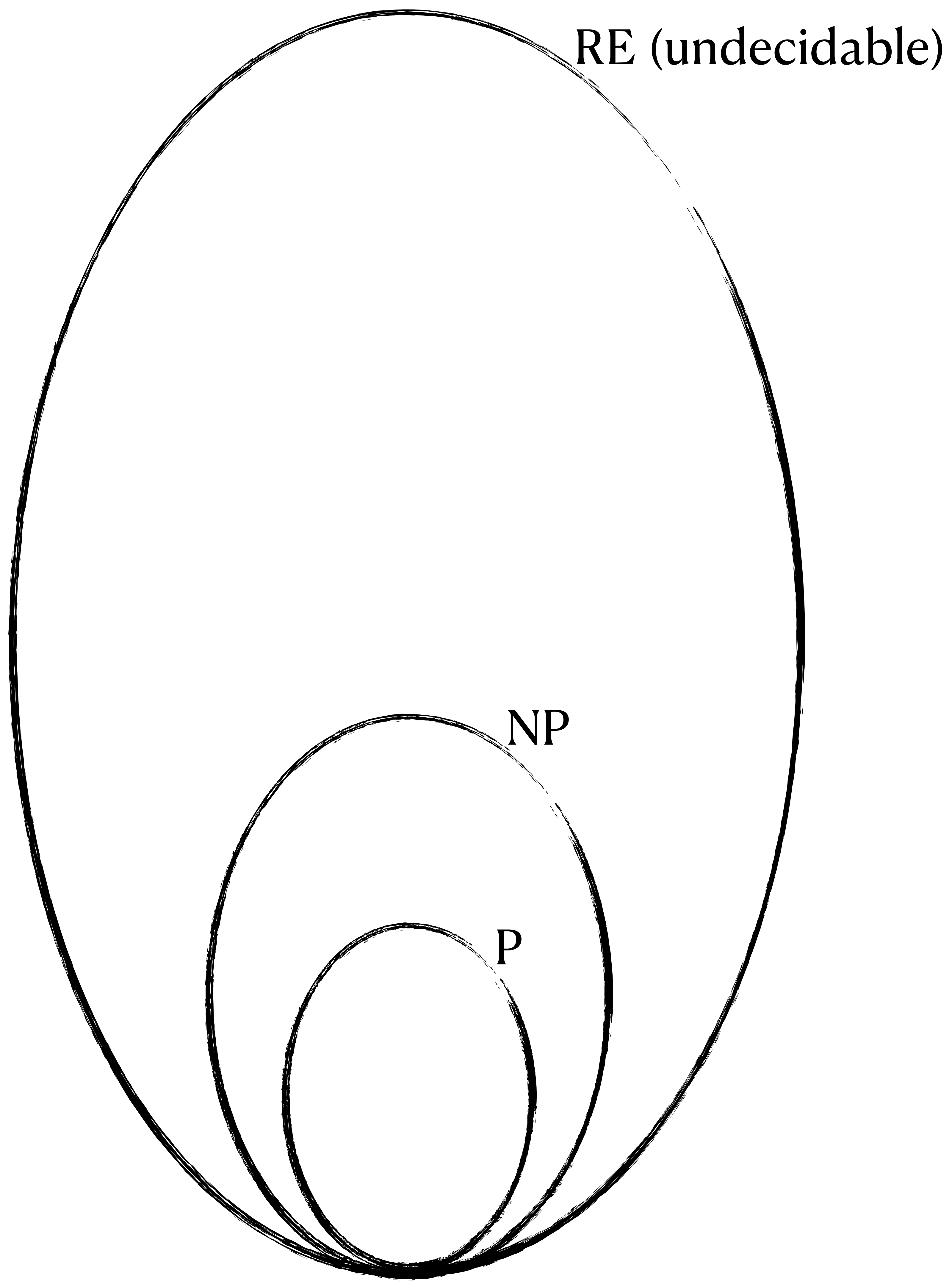
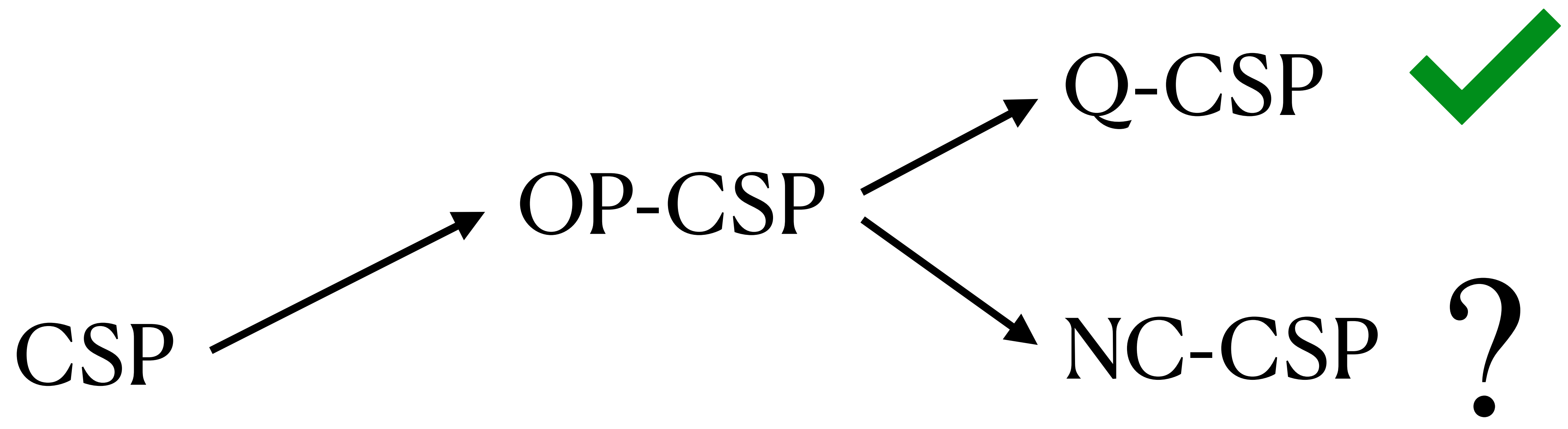




What is remained to do for OP-CSPs?

Quantum generalizations of CSPs



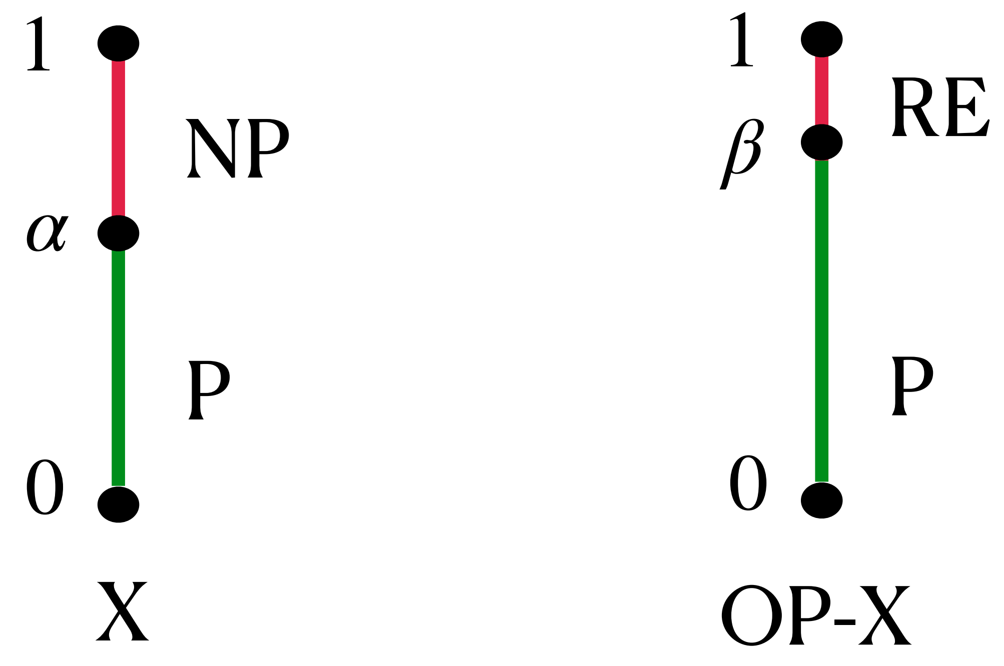
OP-CSP is short for operator-CSP

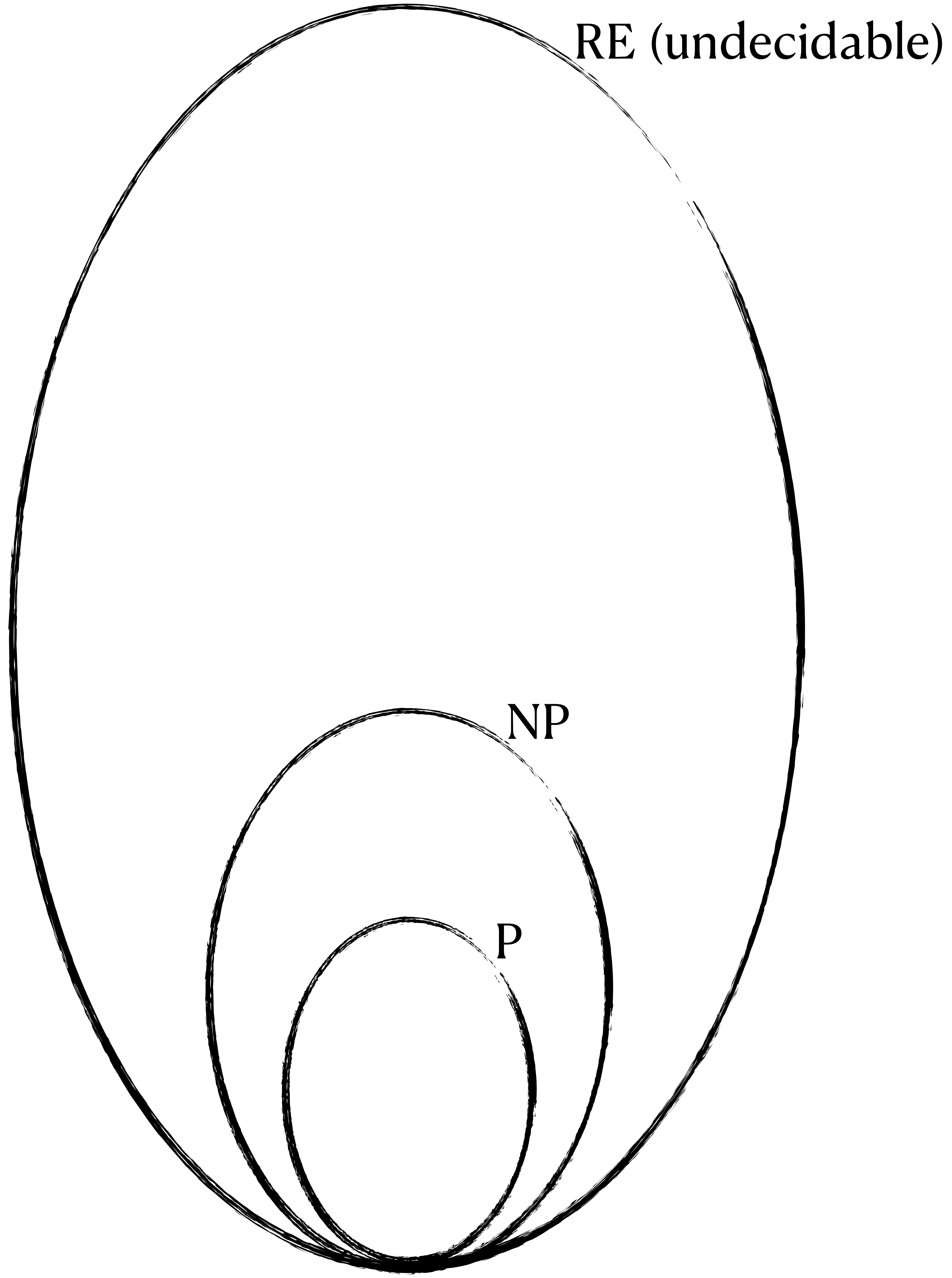
Q-CSP is short for quantum-CSP

NC-CSP is short for noncommutative-CSP

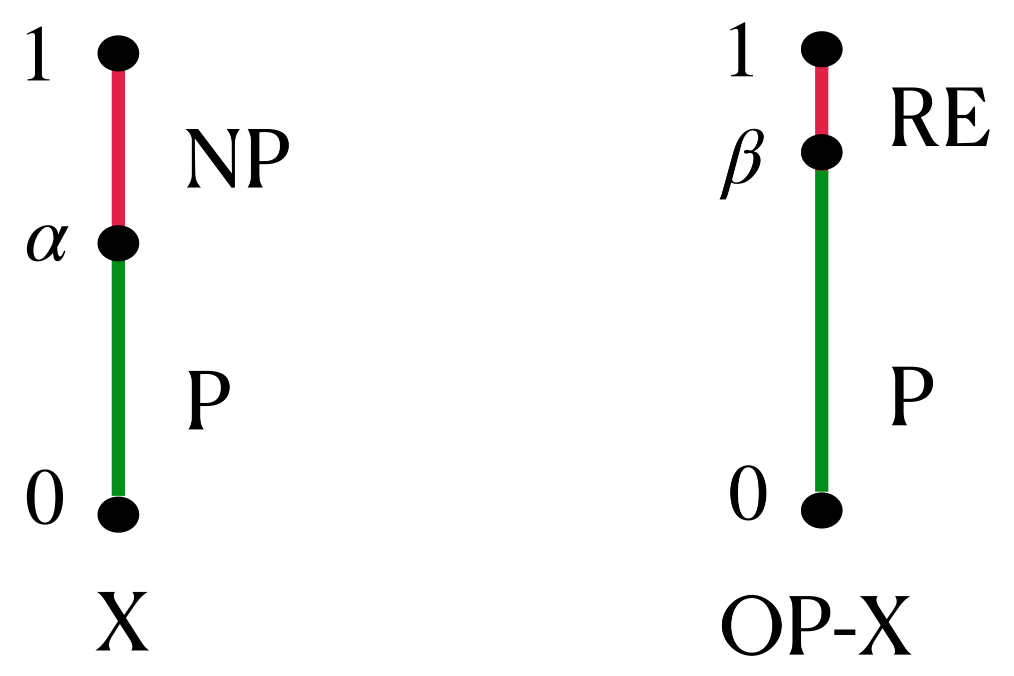
And what is the outlook on the future?

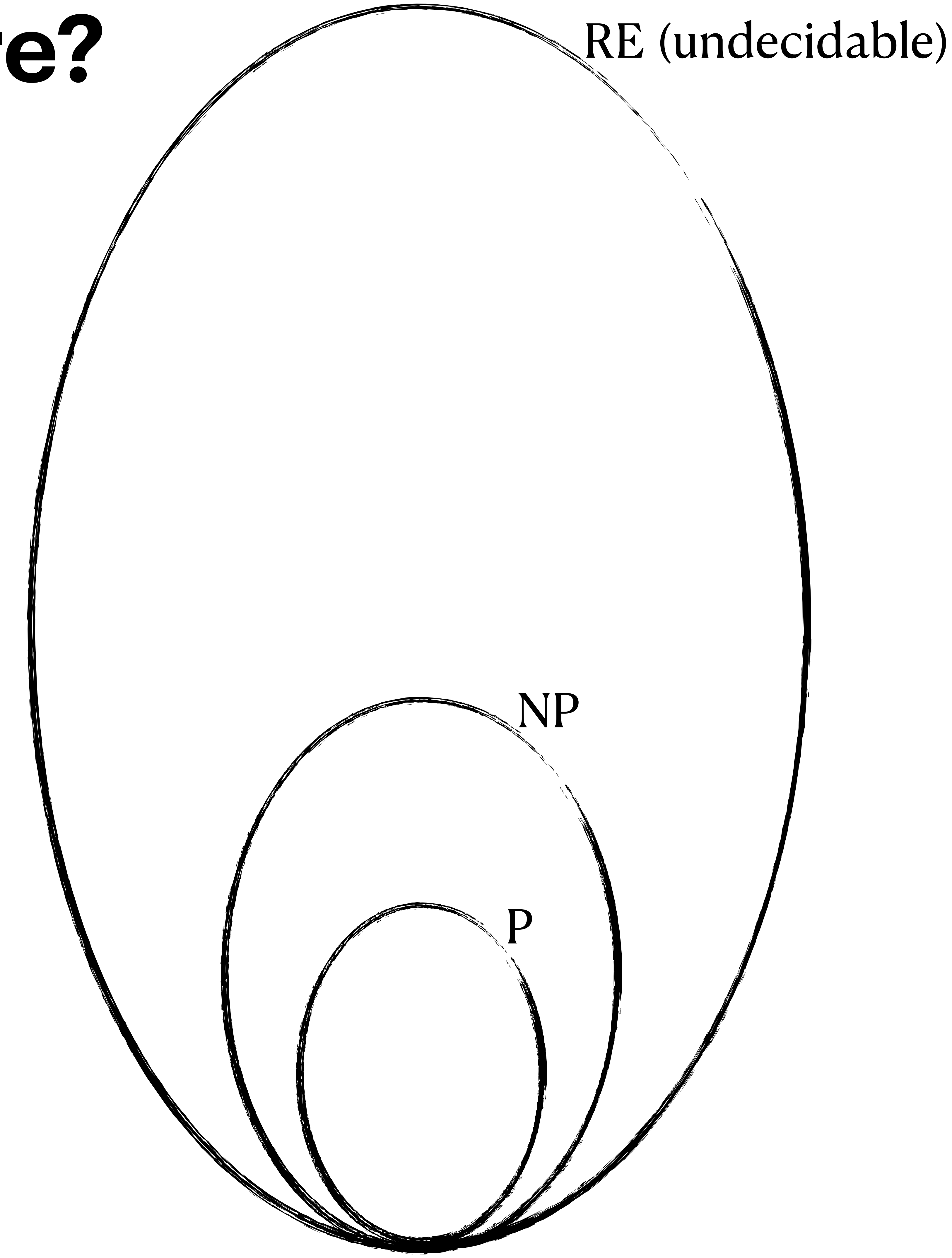
Complexity Landscape





Complexity Landscape: But why quantum classes are not present in this picture?





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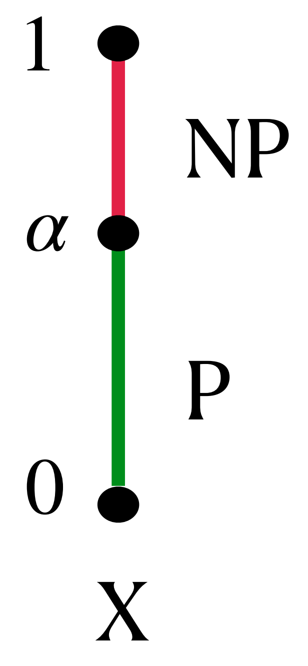

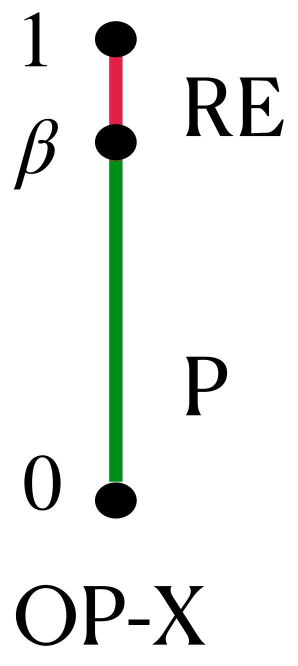

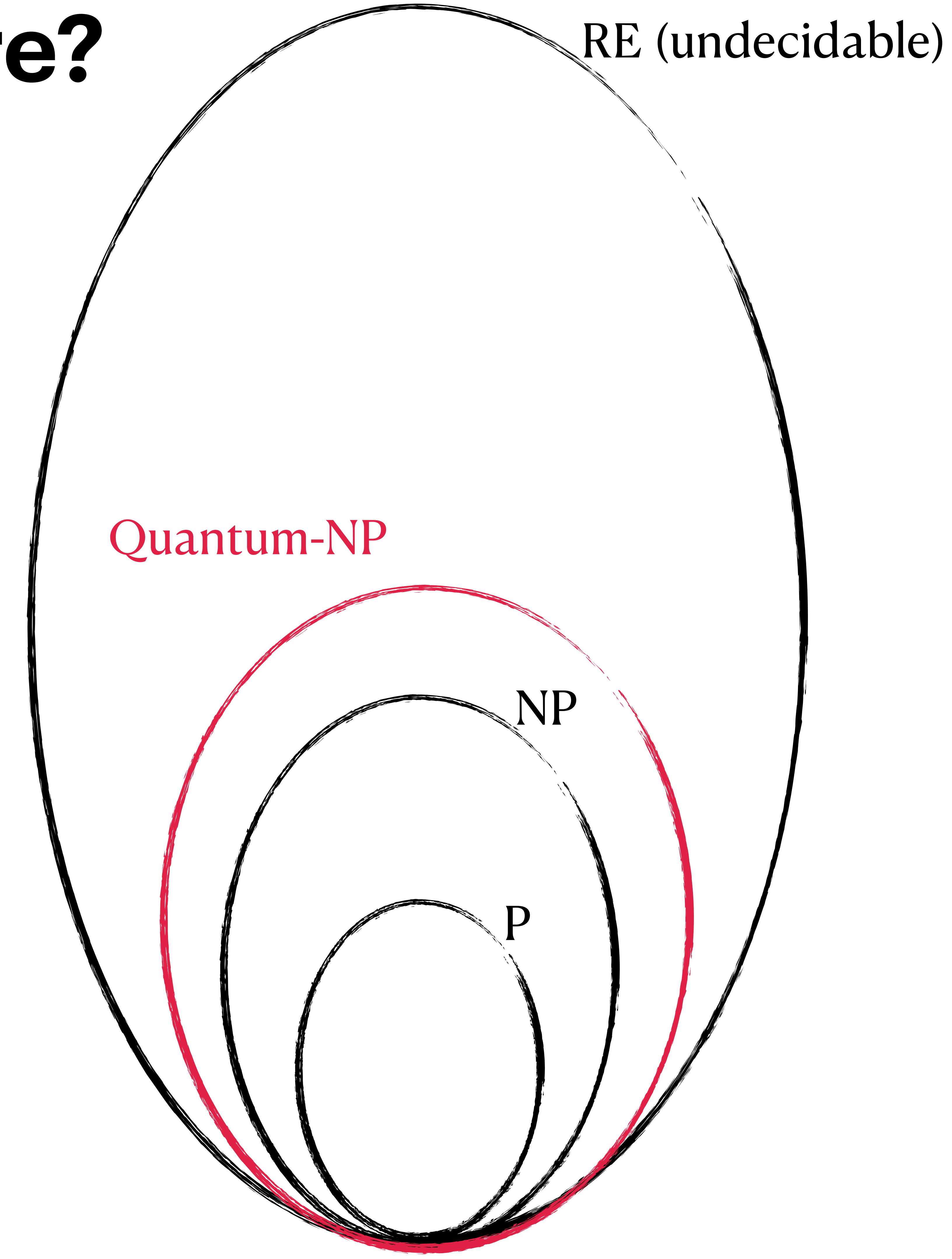

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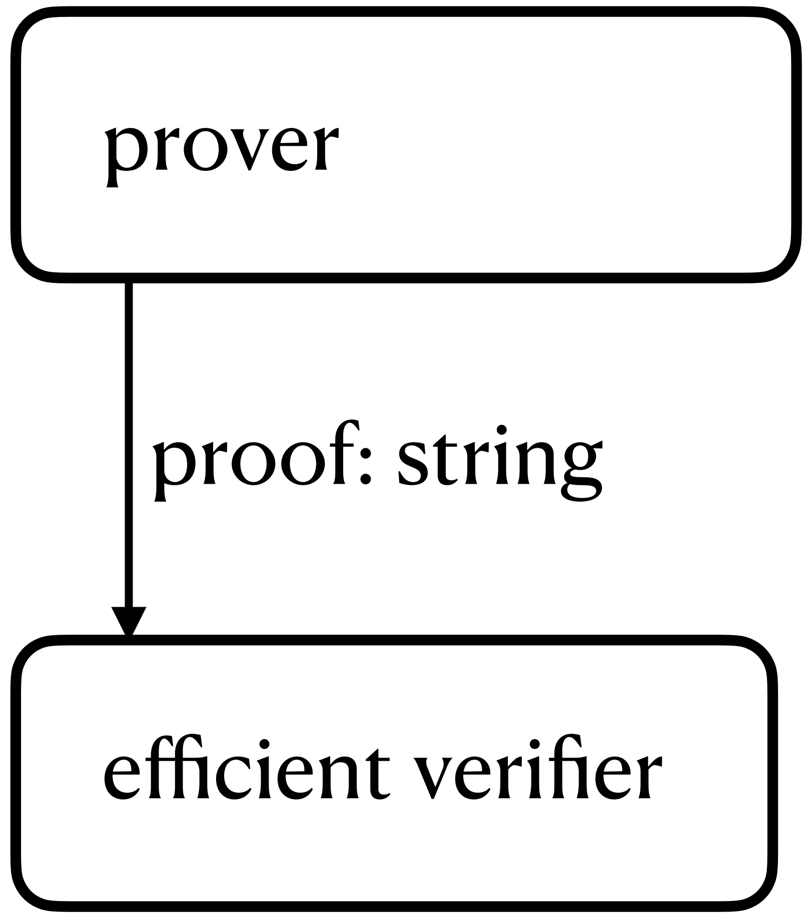
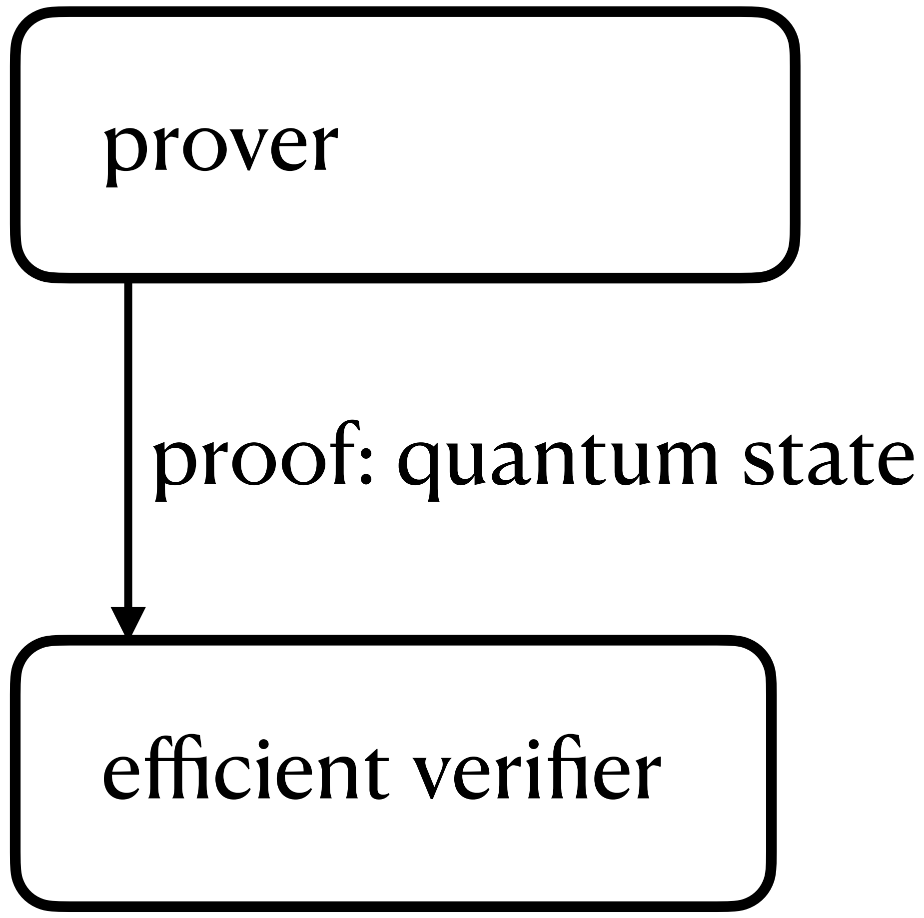
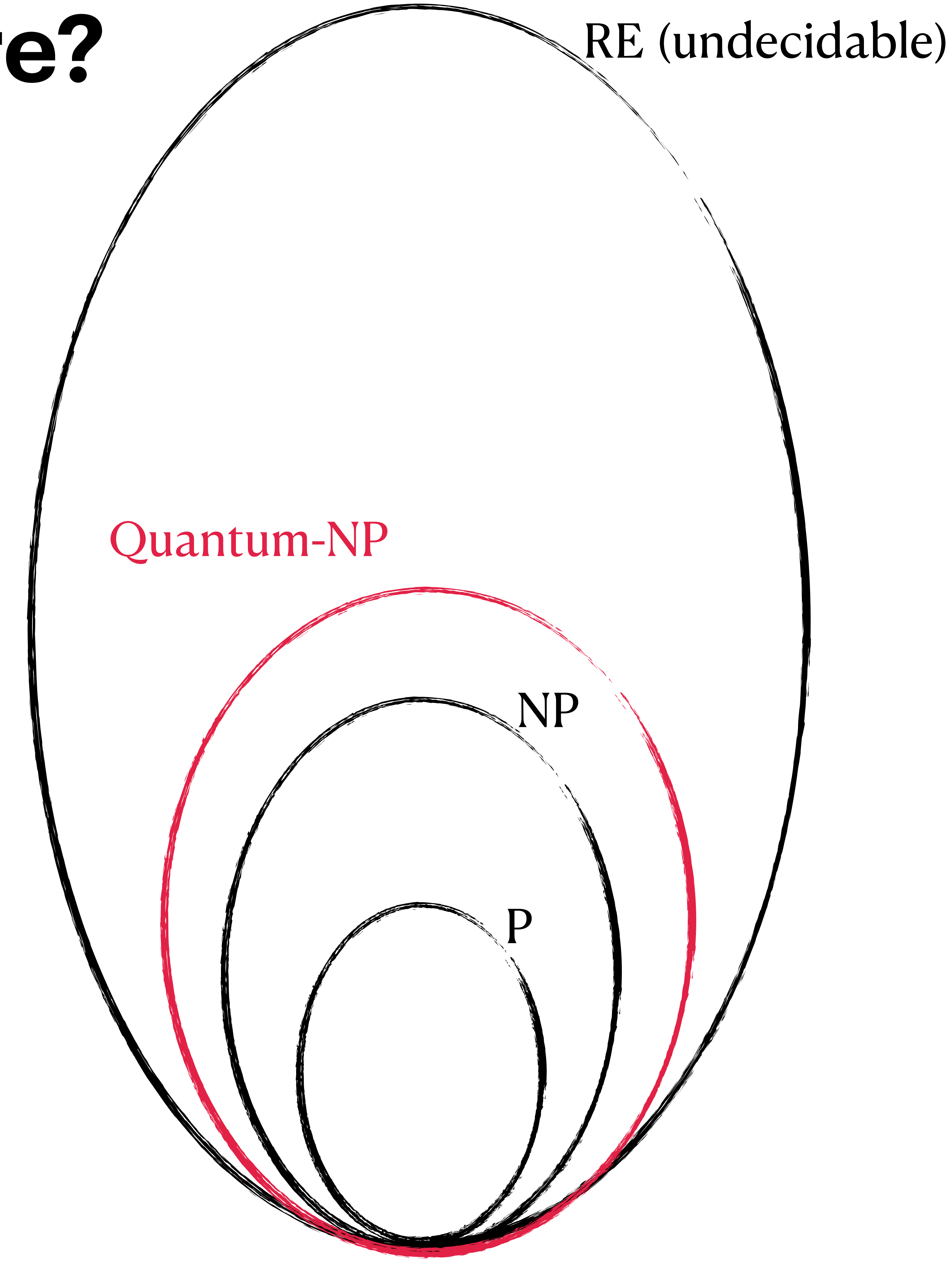
Quantum-NP is also known as QMA.
Recall the **proof verification definition** of NP?
Proof is a **string** in that definition.
If you allow the proof to be a **quantum state**, then you arrive at the definition of Quantum-NP.

NP:



Quantum NP:





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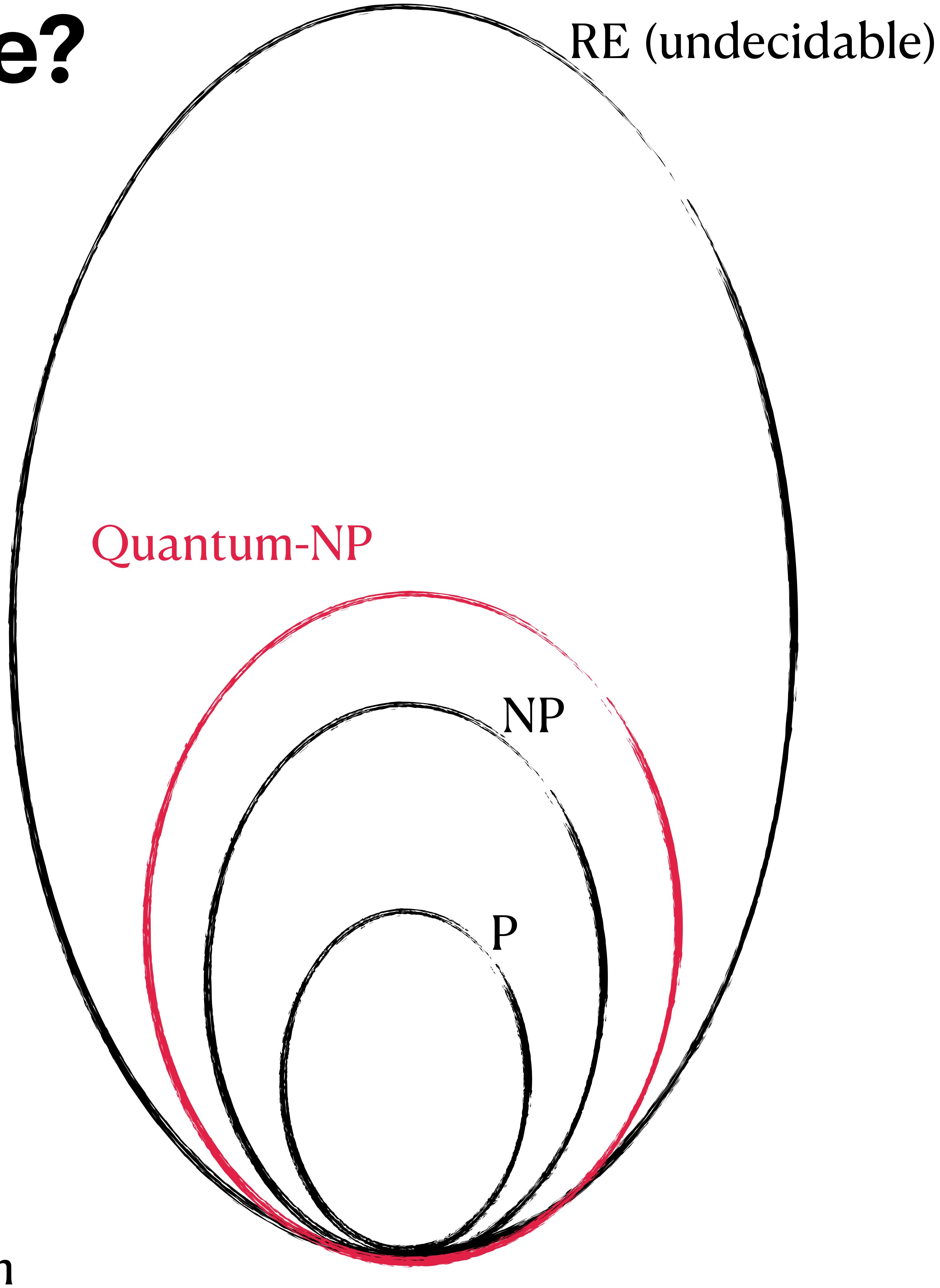

In fact, there is a natural variant of operator CSPs that falls in Quantum-NP.

And this could improve our understanding of this complexity class (and quantum computing as consequence).

Main Part

MagicSquare

MagicSquare

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

x_{ij} are ± 1

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This is a consequence of commutativity.

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Operator Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
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Pauli matrices:

$$I = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad X = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \quad Y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}, \quad Z = \begin{bmatrix} 1 & 0 \\ 0 & -1 \end{bmatrix}.$$

$\otimes$ is called the Kronecker products.

For example $I \otimes X$ is the matrix $\begin{bmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 1 & 0 \end{bmatrix}$.

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Our matrices are unitary operators: $U^*U = I$.

A unitary operators has complex eigenvalues with an absolute value of 1.

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Our matrices are unitary operators with two eigenvalues ± 1.

Such matrices are called binary observables. They model quantum measurements with binary outcomes.

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Mermin 1990 and Peres 1990

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$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Our matrices are unitary operators with two eigenvalues ± 1.

Such matrices are called binary observables. They model quantum measurements with binary outcomes.

Think of them as some generalization of binary random variables with some strangeness sprinkled on top:

probability theory ---> quantum probability theory

expectation ---> trace

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

x_{ij} are ± 1

Operator Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

These are binary observables (unitaries with ± 1 eigenvalues)

MagicSquare

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
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Commutation relations: Pair of matrices sharing a row or column commute.

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These are binary observables (unitaries with ± 1 eigenvalues)

Commutation relations: Pair of matrices sharing a row or column commute.

Quantum measurement destroys (collapses) the state of the system.

So the order of measurement is very crucial.

But, when two observables commute, the order of measurement does not matter.

MagicSquare

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
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These are binary observables (unitaries with ± 1 eigenvalues), and they satisfy the row and column commutation relations

MagicSquare

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Has no solution

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X_{ij} are binary observables (unitaries with ± 1 eigenvalues),
and satisfy the row and column commutation relations

Has a unique solution

A bit more formally: In every solution, every off-diagonal pair of observables must anticommute. That is for example $X_{21}X_{12} = -X_{12}X_{21}$.

And every two anticommuting observables are isometrically equivalent to Pauli operators X and Z.

Why should we care about the MagicSquare and its operator variant?

MagicSquare as a nonlocal game

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

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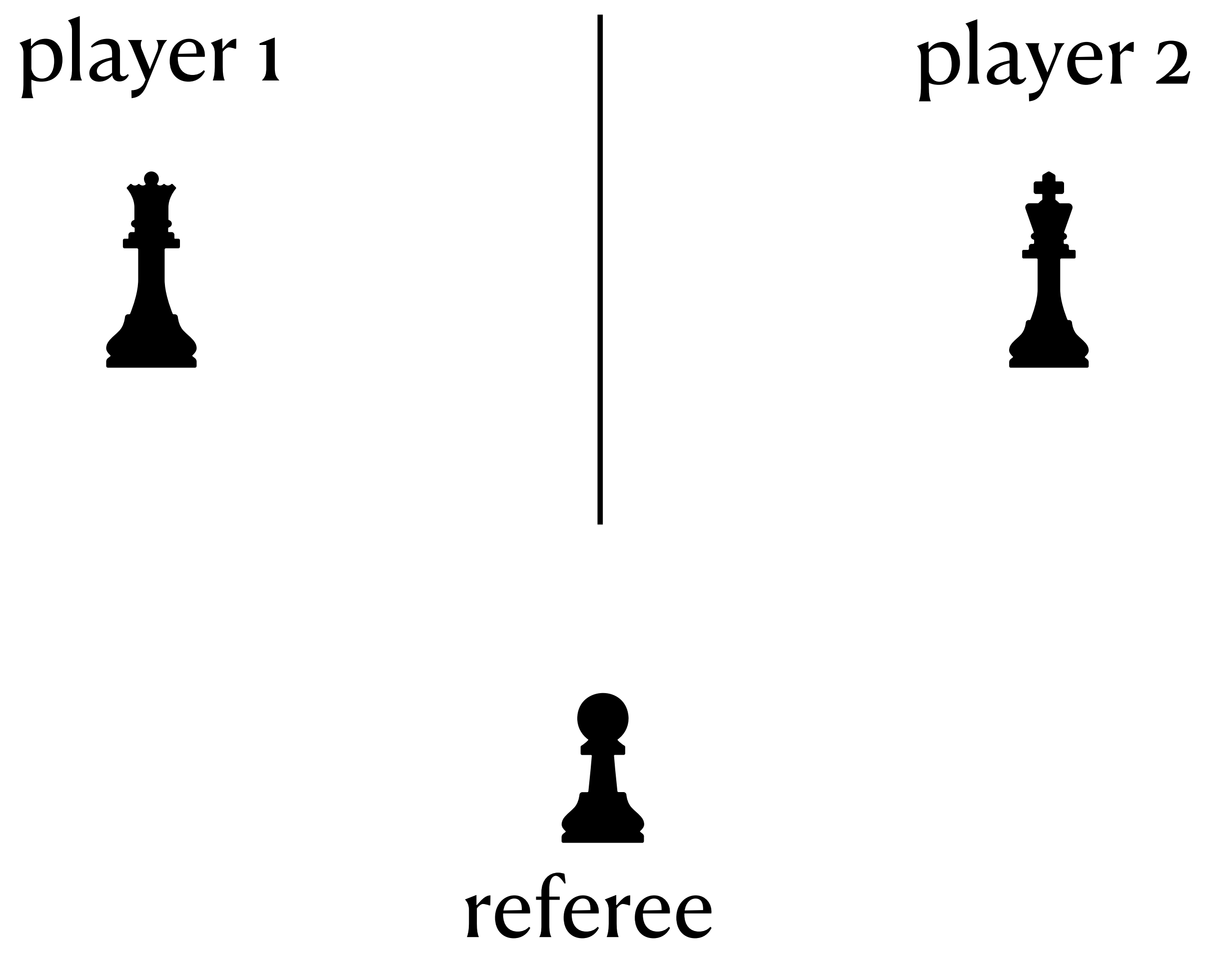

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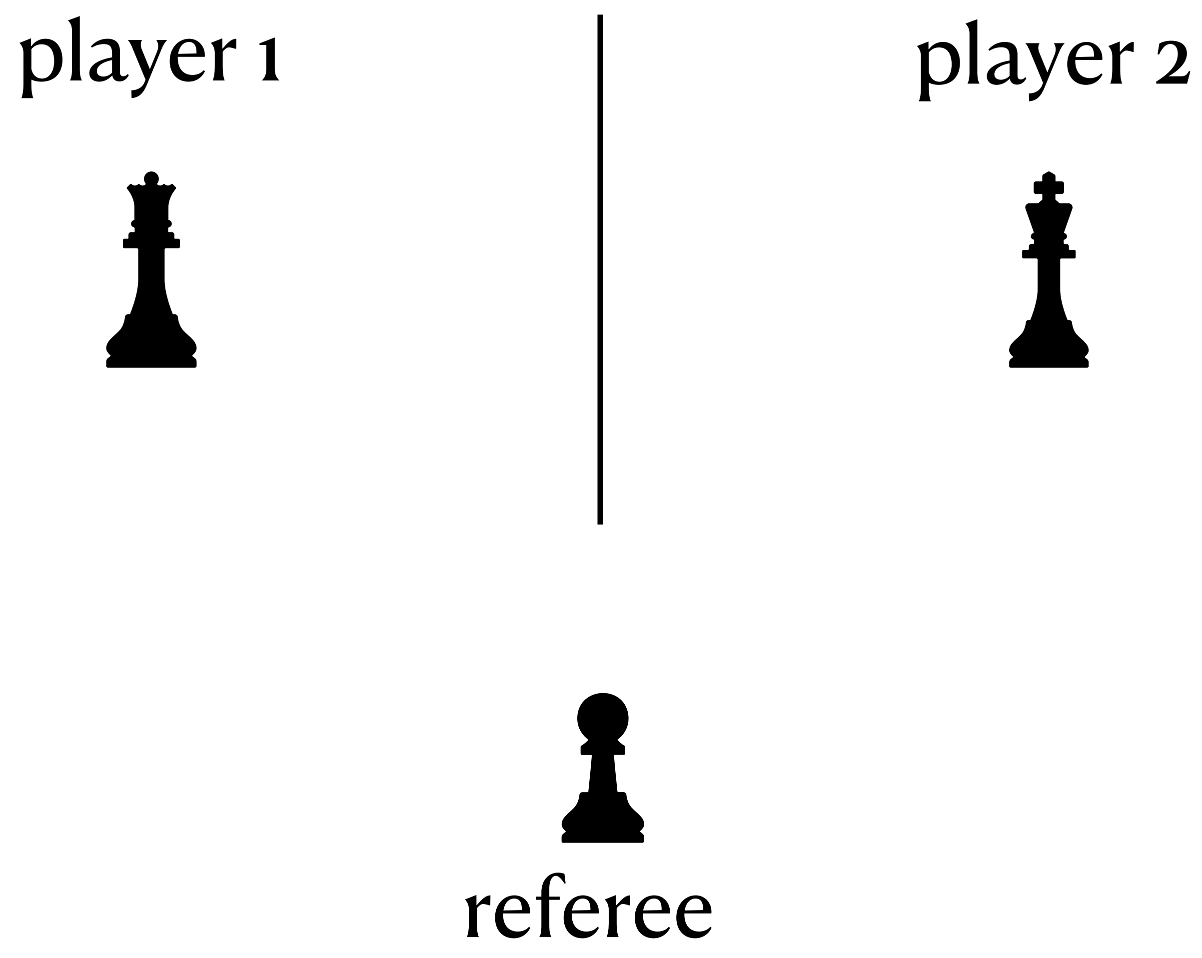
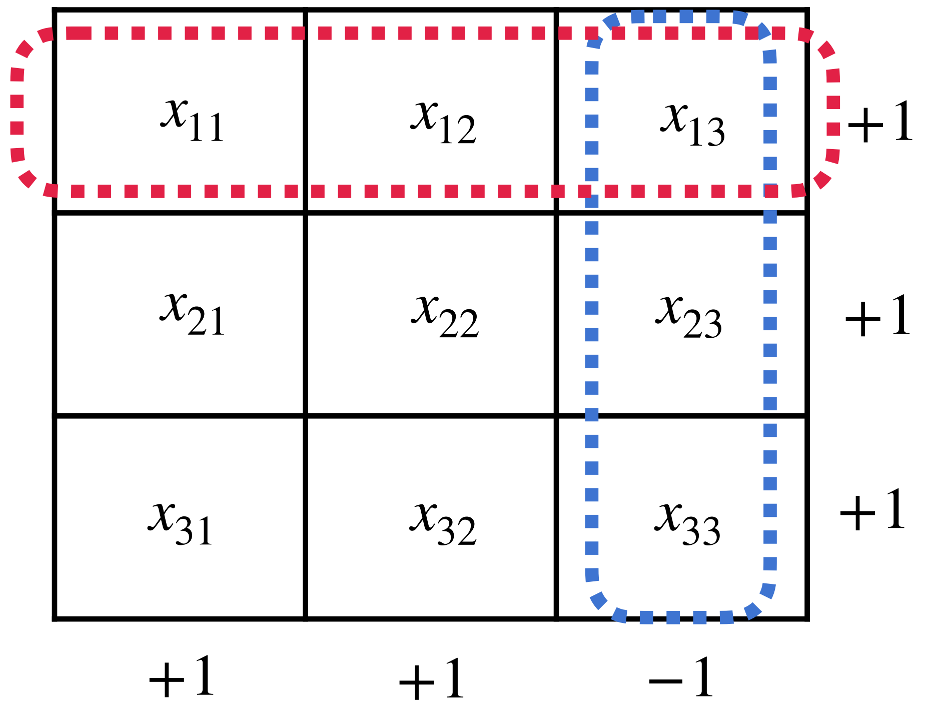
x_{ij} are ± 1

1. Referee chooses a row and a column and sends them to player 1 and player 2, respectively.



MagicSquare as a nonlocal game



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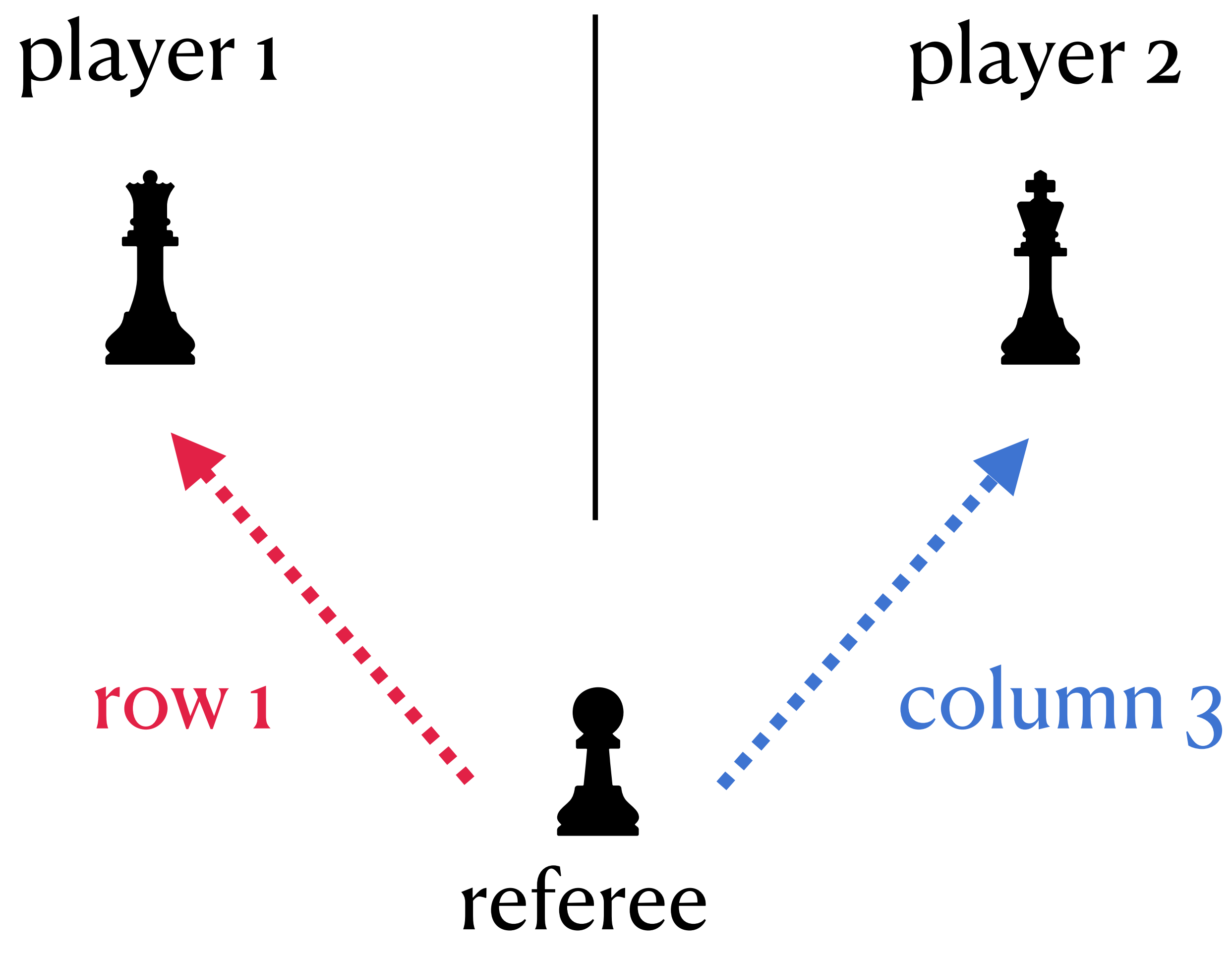

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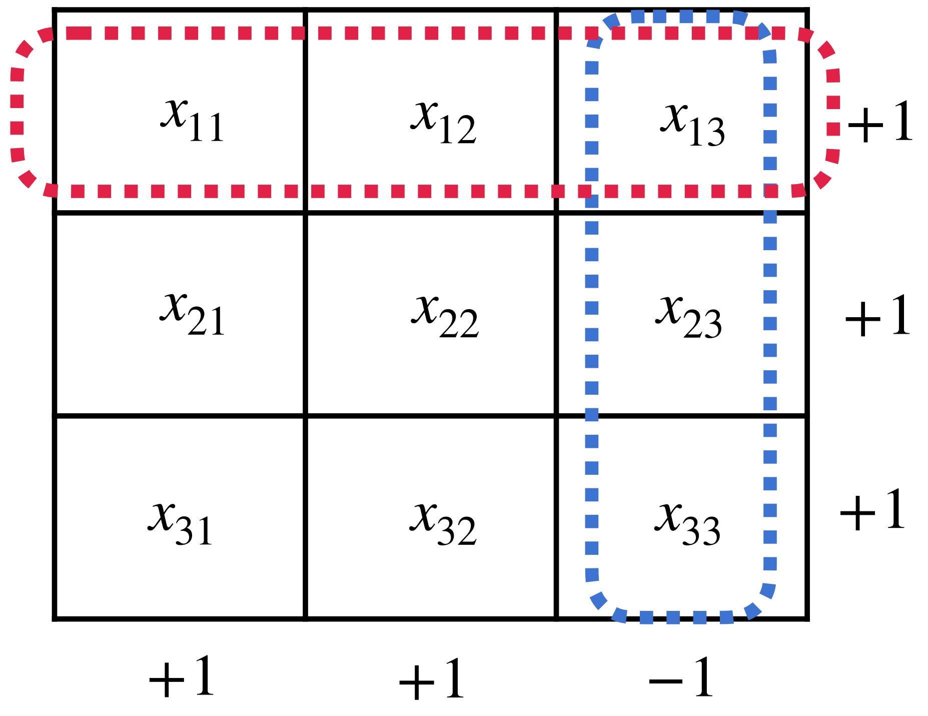
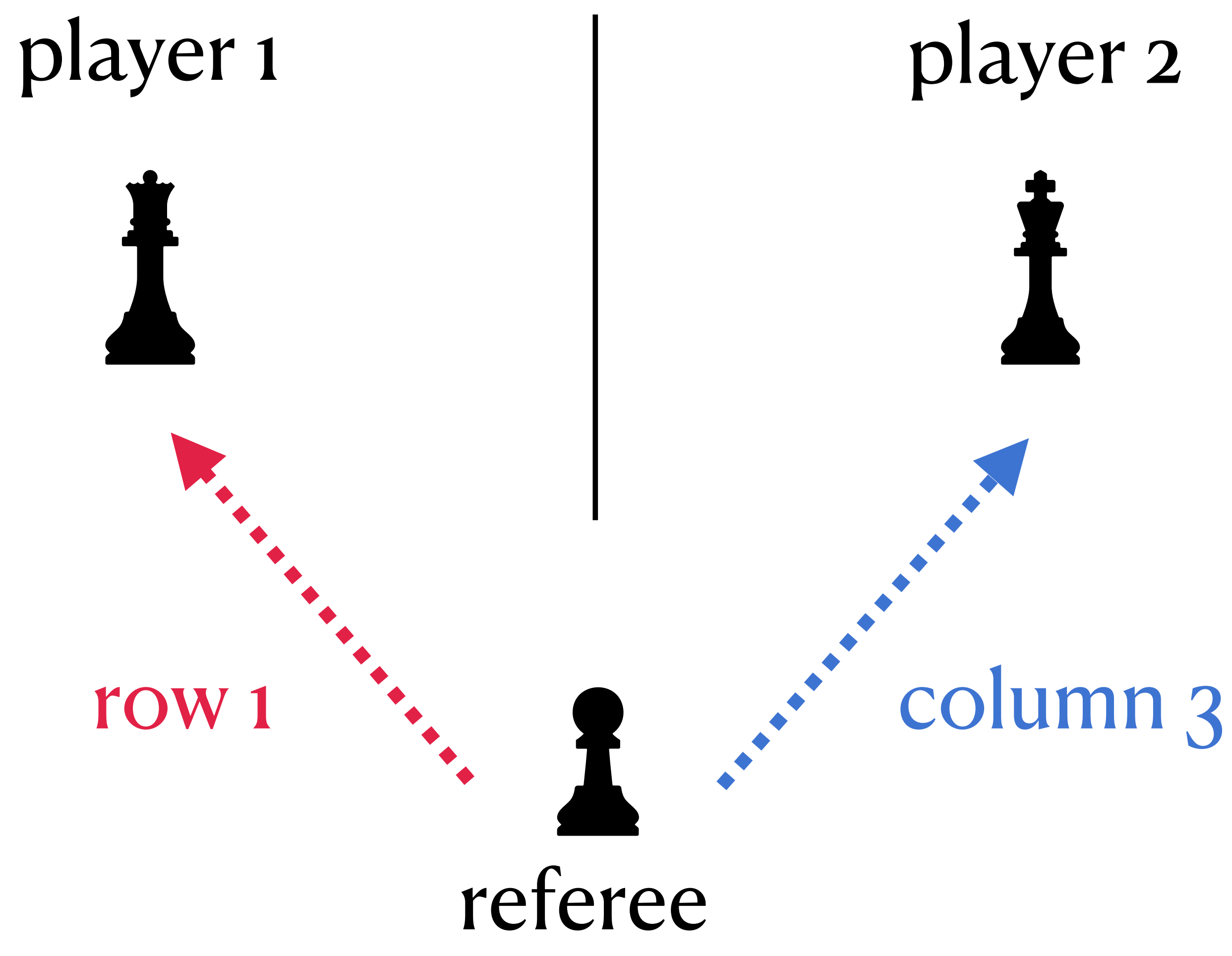


x_{ij} are ± 1

1. Referee chooses a row and a column and sends them to player 1 and player 2, respectively.
2. Players respond with an assignment to the variables in their row or column.



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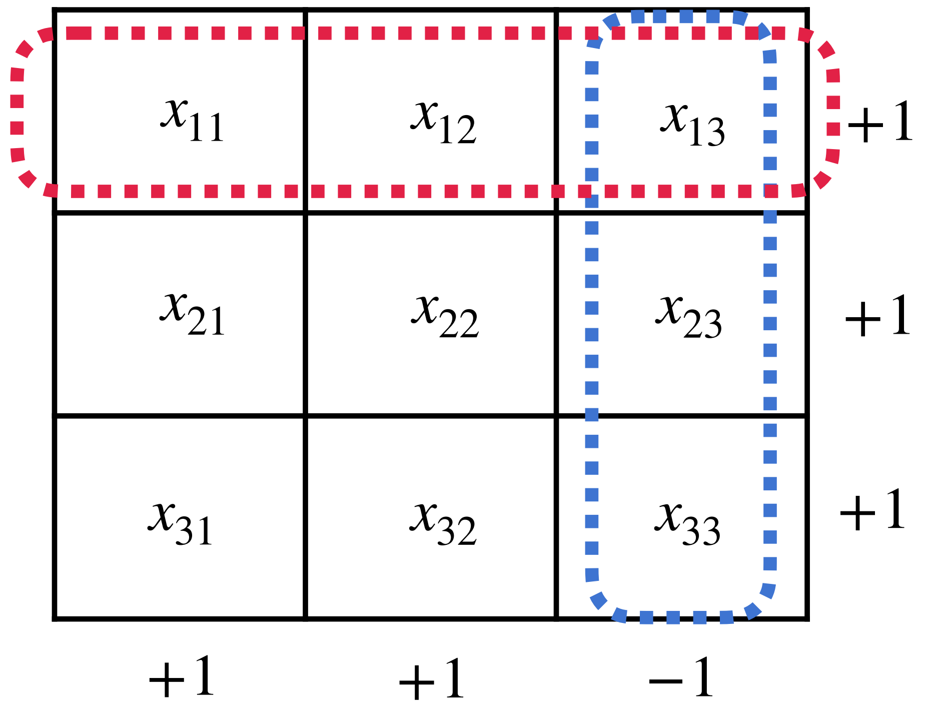

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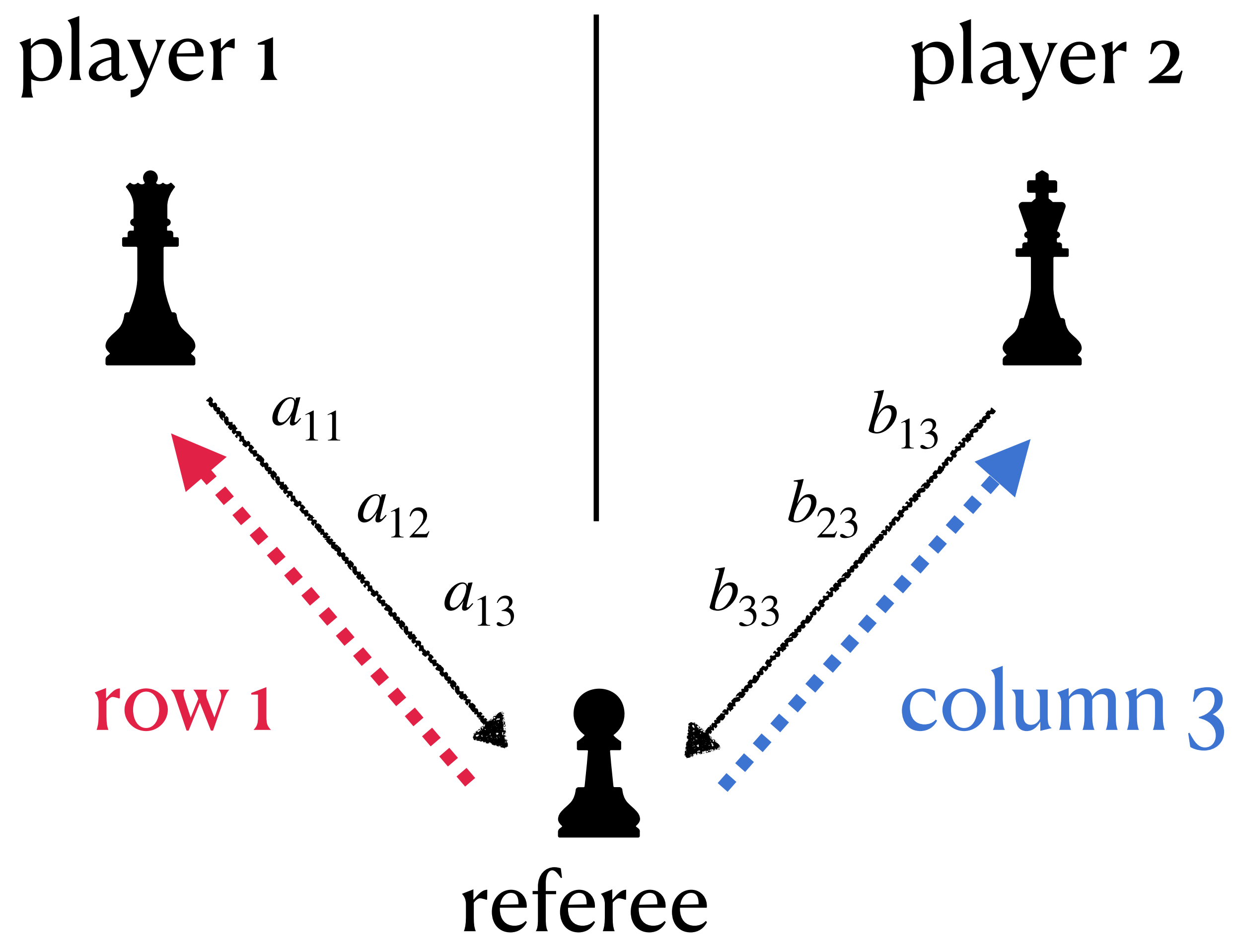
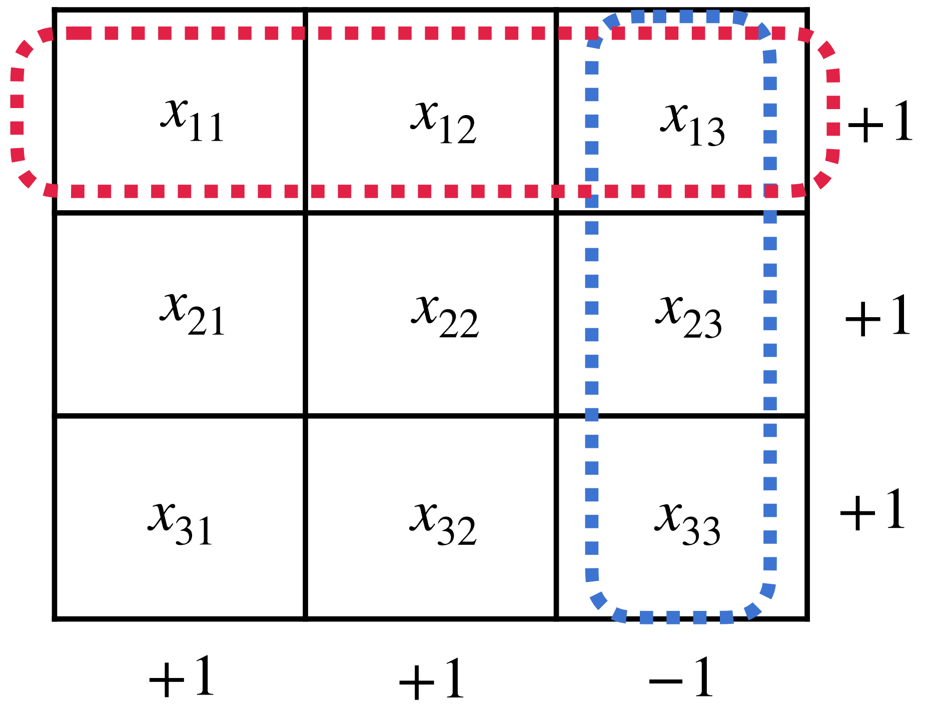


MagicSquare as a nonlocal game



x_{ij} are ± 1

1. Referee chooses a row and a column and sends them to player 1 and player 2, respectively.
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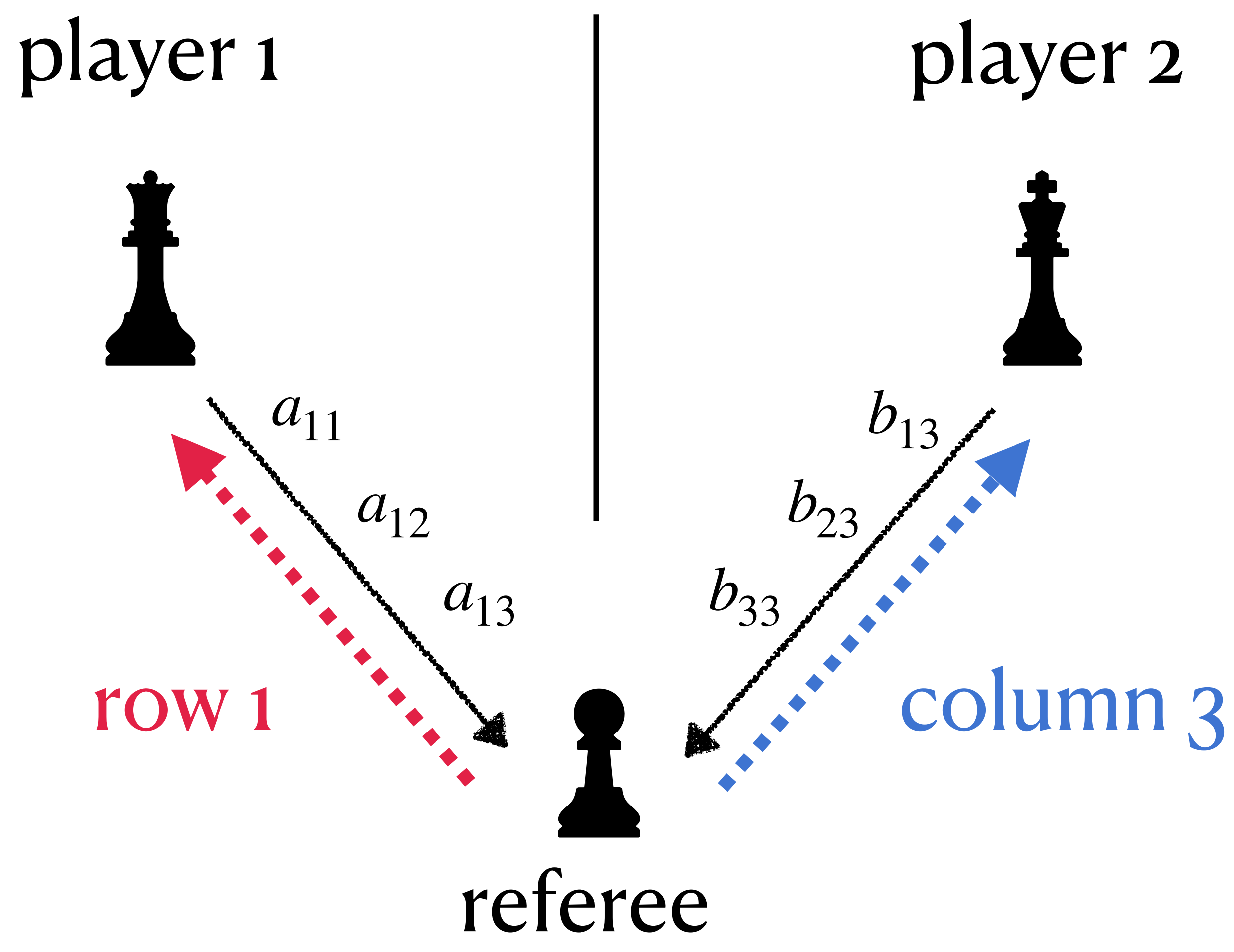
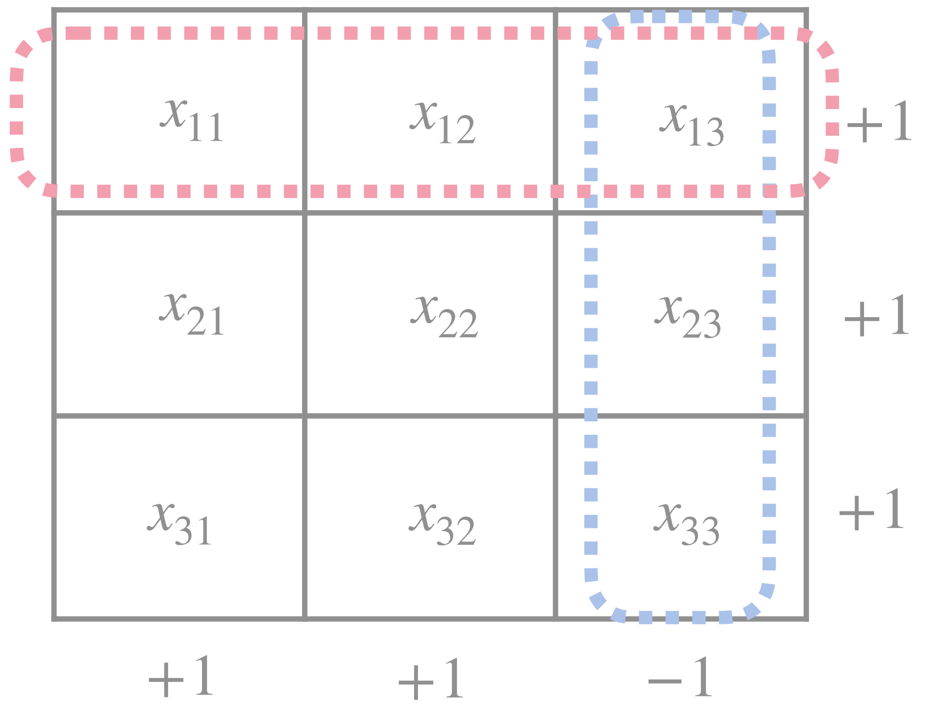
3. Winning conditions:
    A. Satisfy the row and column constraints
    B. Be consistent



MagicSquare as a nonlocal game



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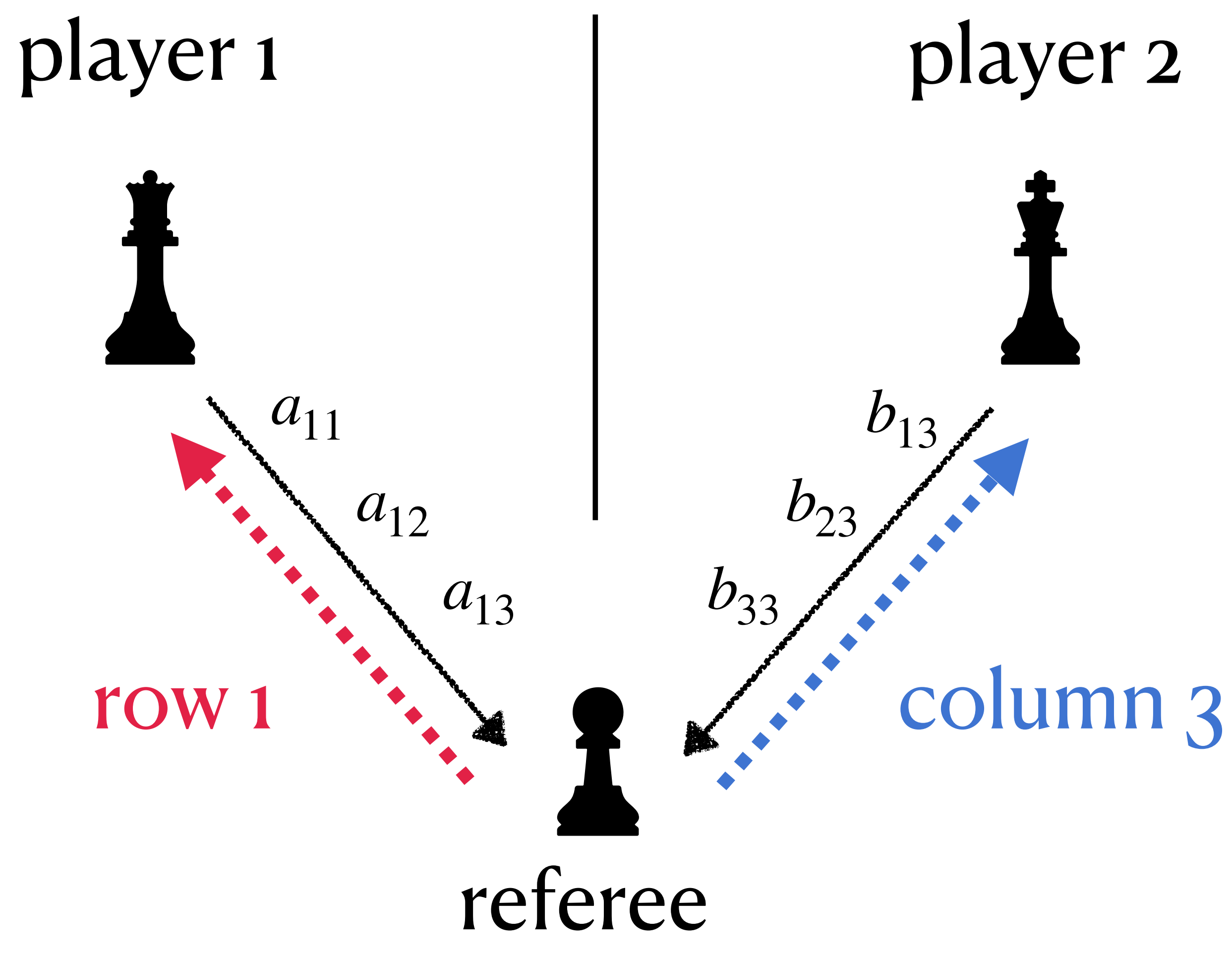
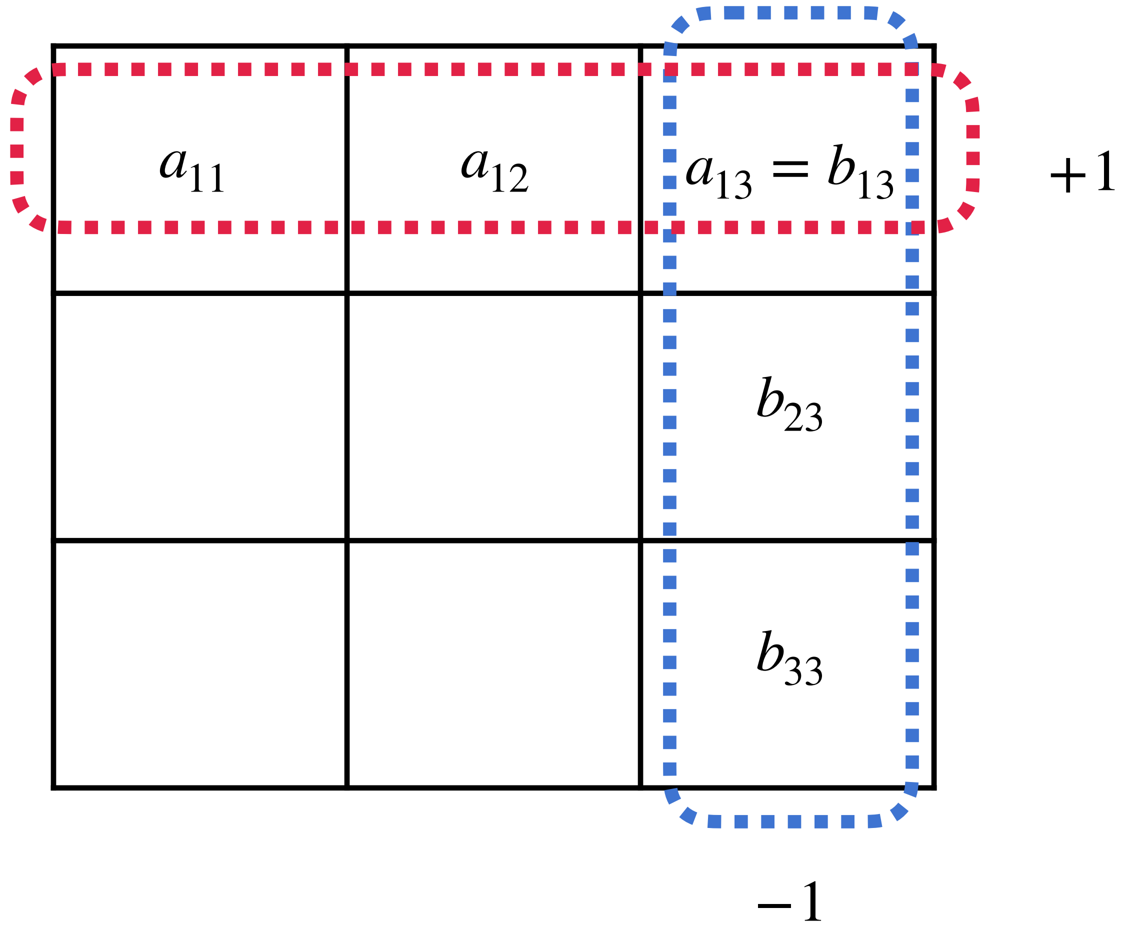
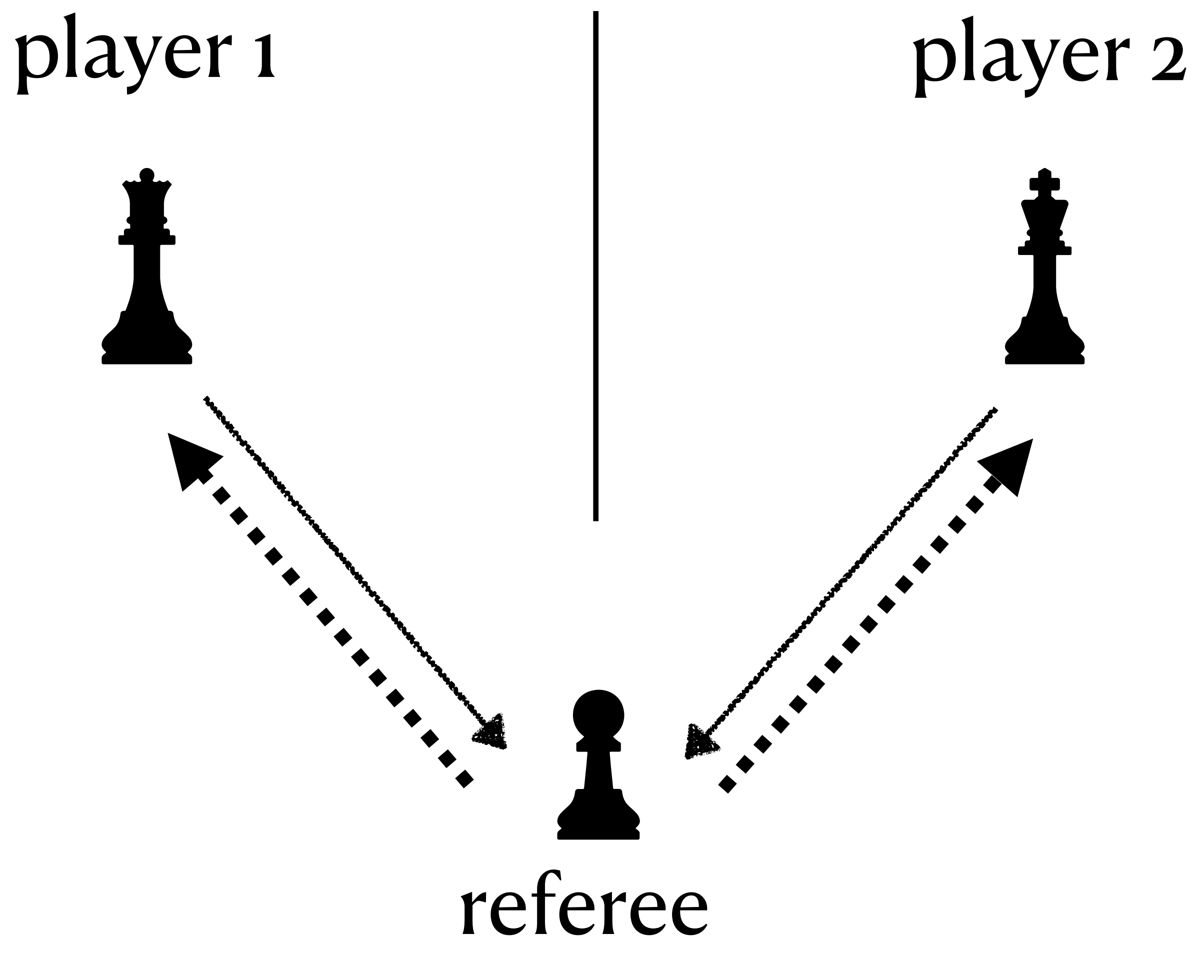


Referee checks the winning conditions:



MagicSquare as a nonlocal game



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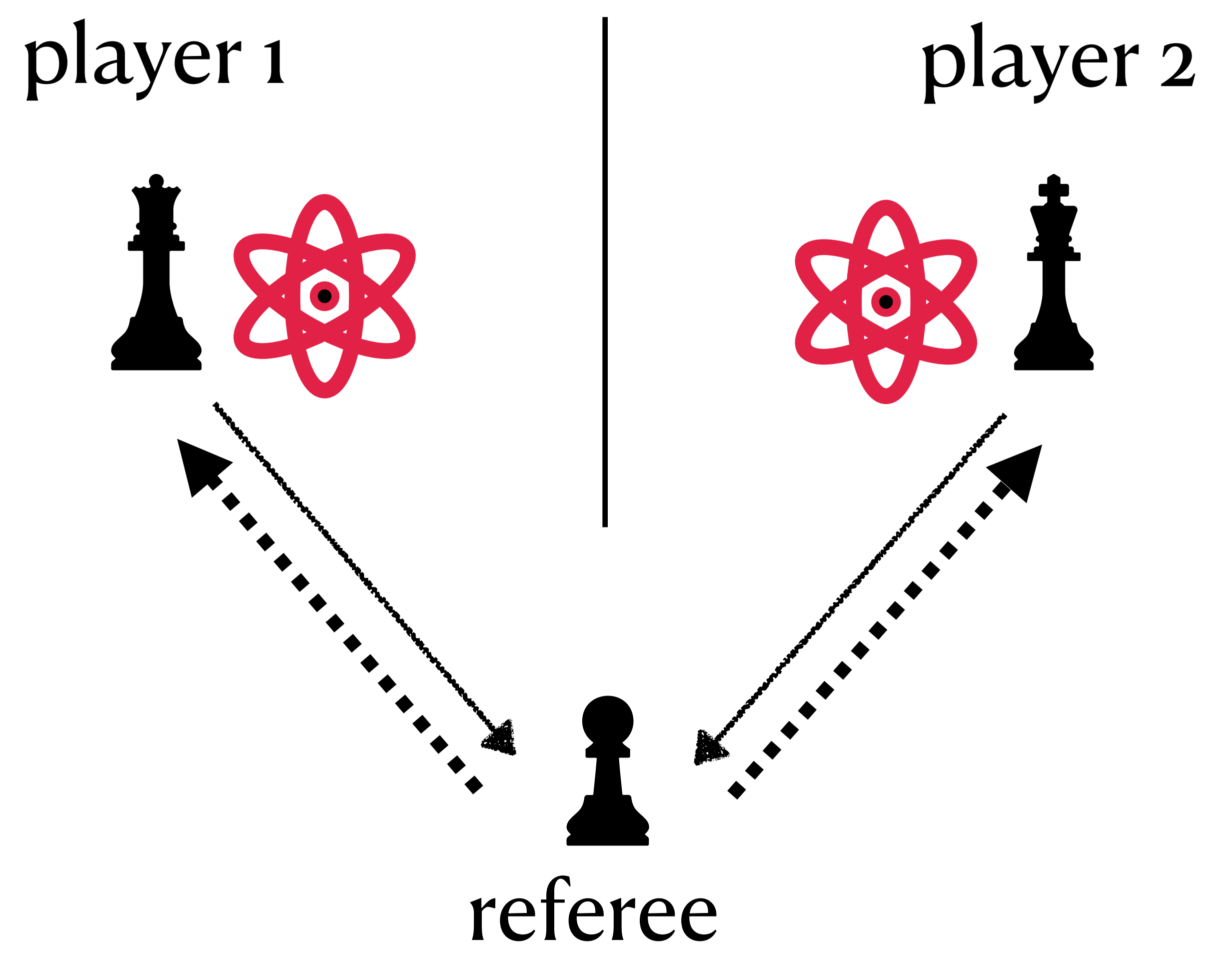
x_{ij} are ± 1

1. Since there is no perfect solution, players cannot win with probability 1.

MagicSquare as a nonlocal game



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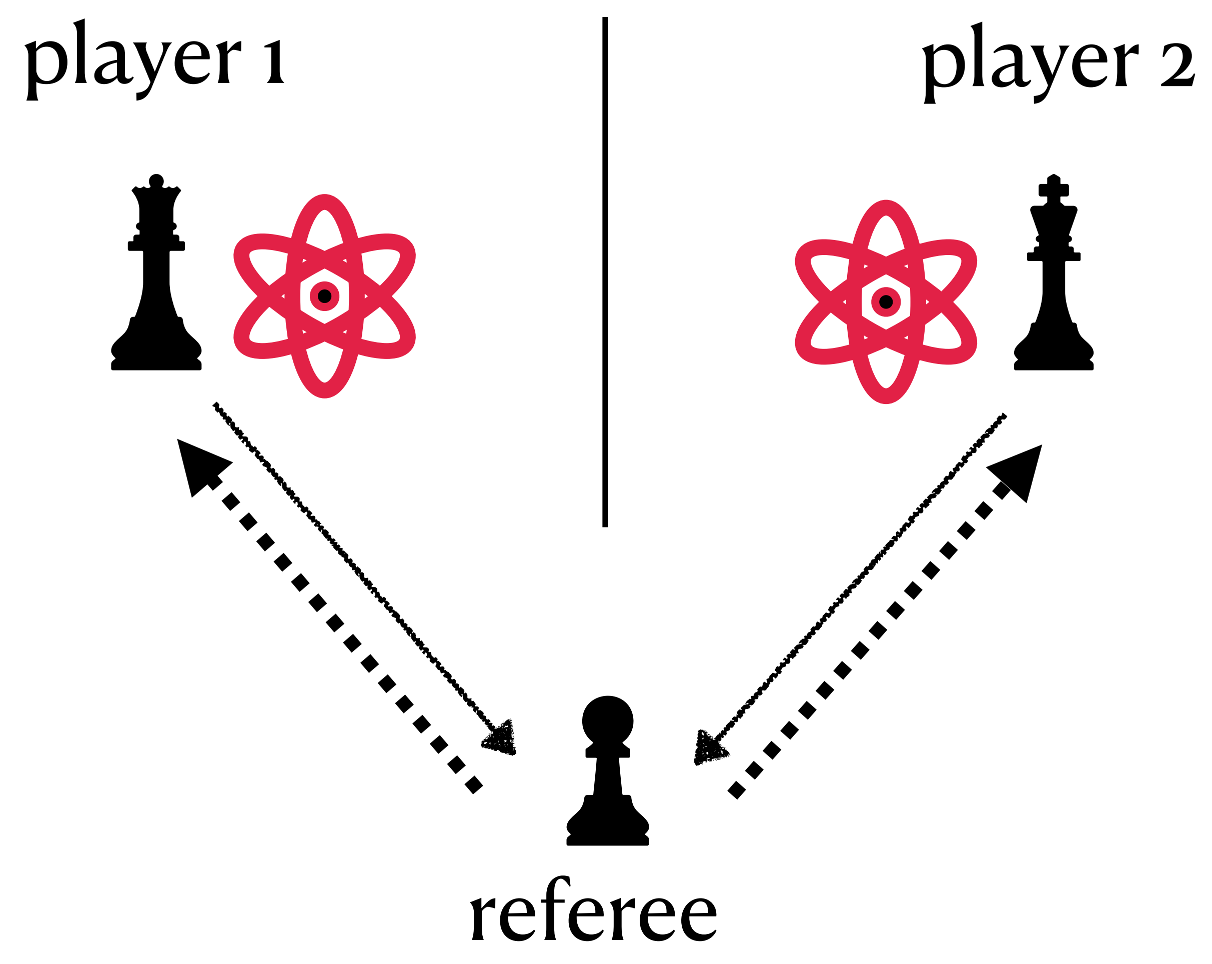
x_{ij} are ± 1

1. Since there is no perfect solution, players cannot win with probability 1.
2. But they can, if they are quantum and they measure using the observables in the operator solution:

$I \otimes X$	$X \otimes I$	$X \otimes X$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$

MagicSquare as a nonlocal game



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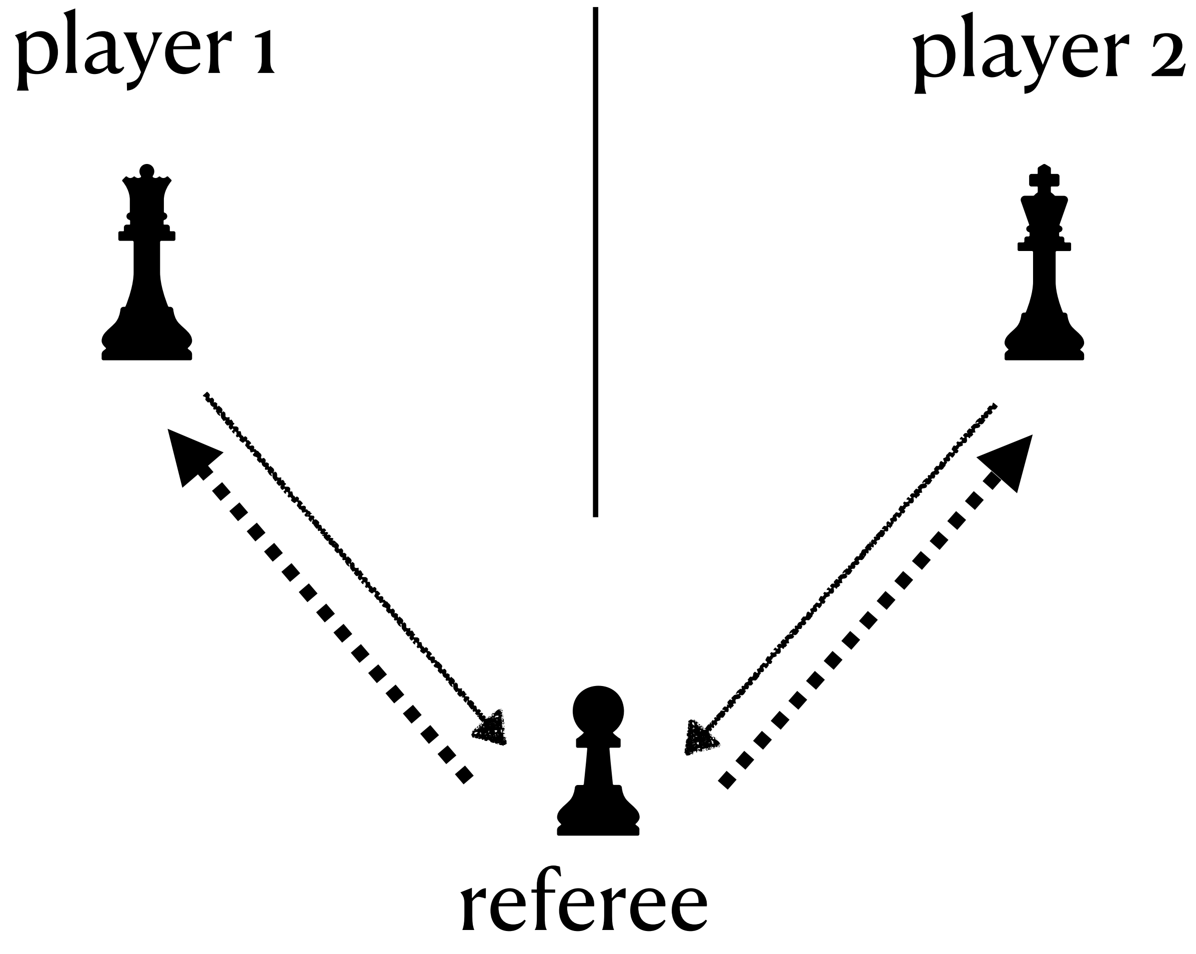
MagicSquare as a nonlocal game



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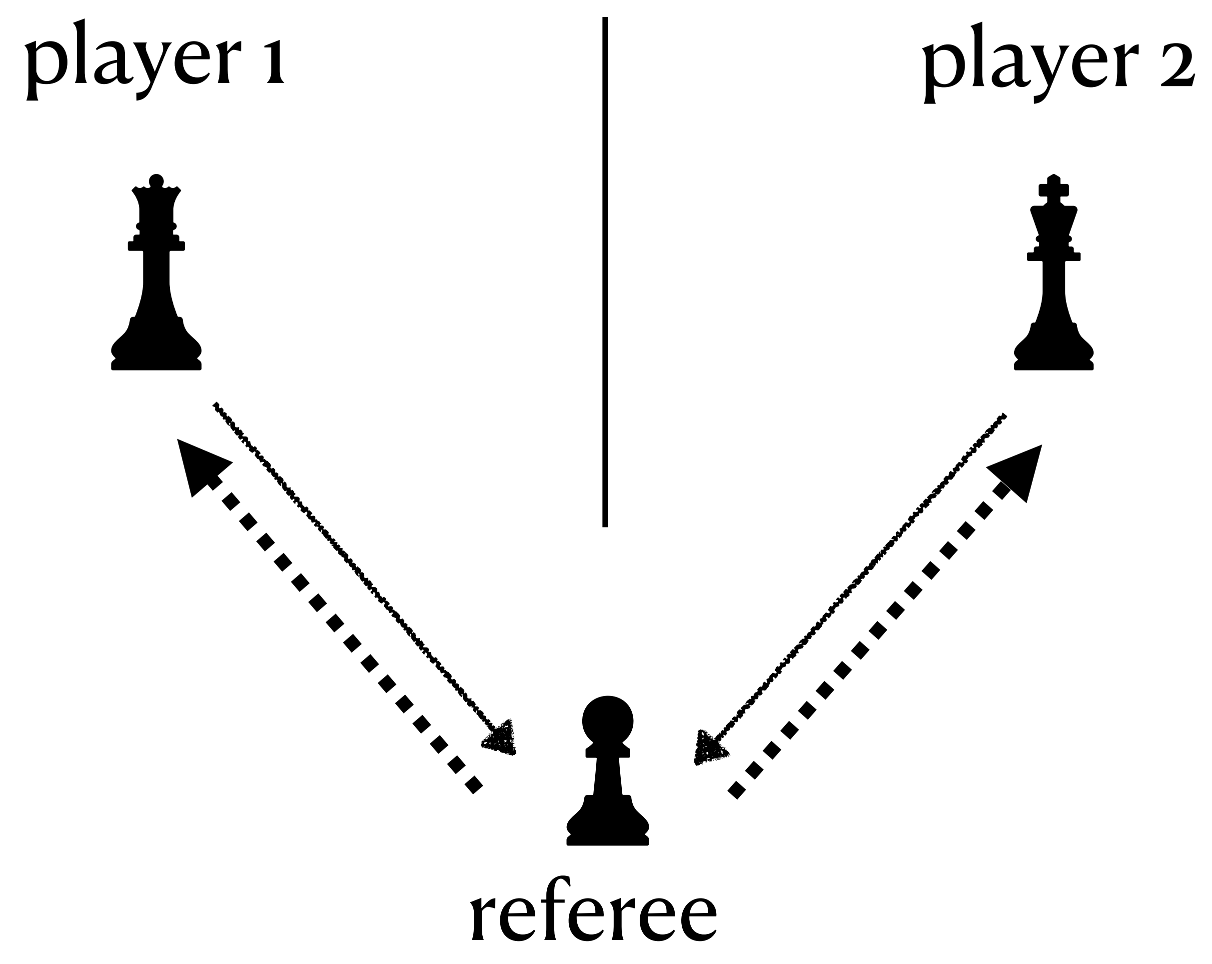
The magic of MagicSquare:

By playing MagicSquare with two players and just observing their winning statistics we can infer

1. whether they are using quantum devices (test of quantum-ness)

MagicSquare as a nonlocal game



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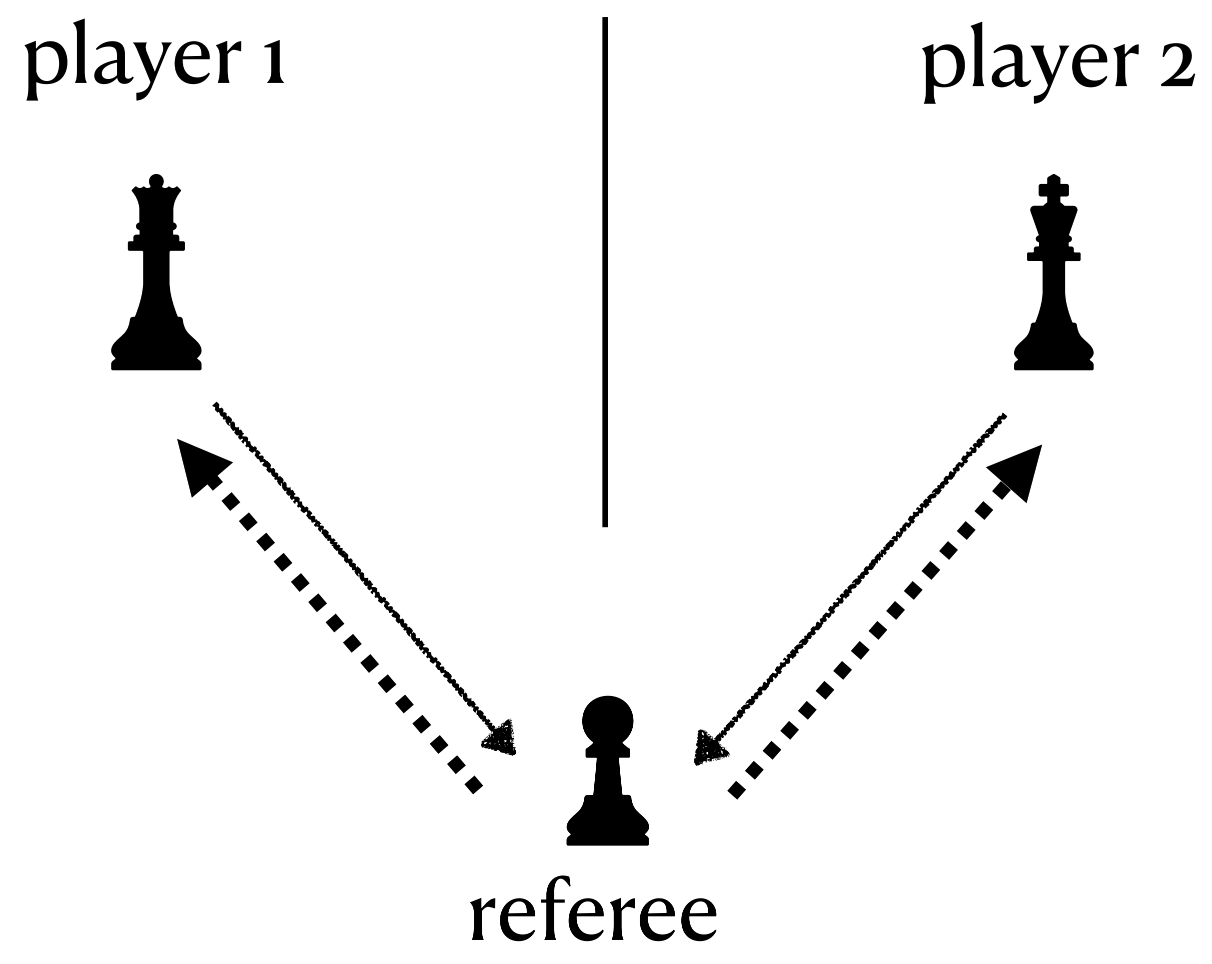
The magic of MagicSquare:

By playing MagicSquare with two players and just observing their winning statistics we can infer

1. whether they are using quantum devices (test of quantum-ness)

2. and if they win all the rounds, the very precise specification of their devices, because of the uniqueness of the operator solution (device-independent cryptography)

MagicSquare as a nonlocal game



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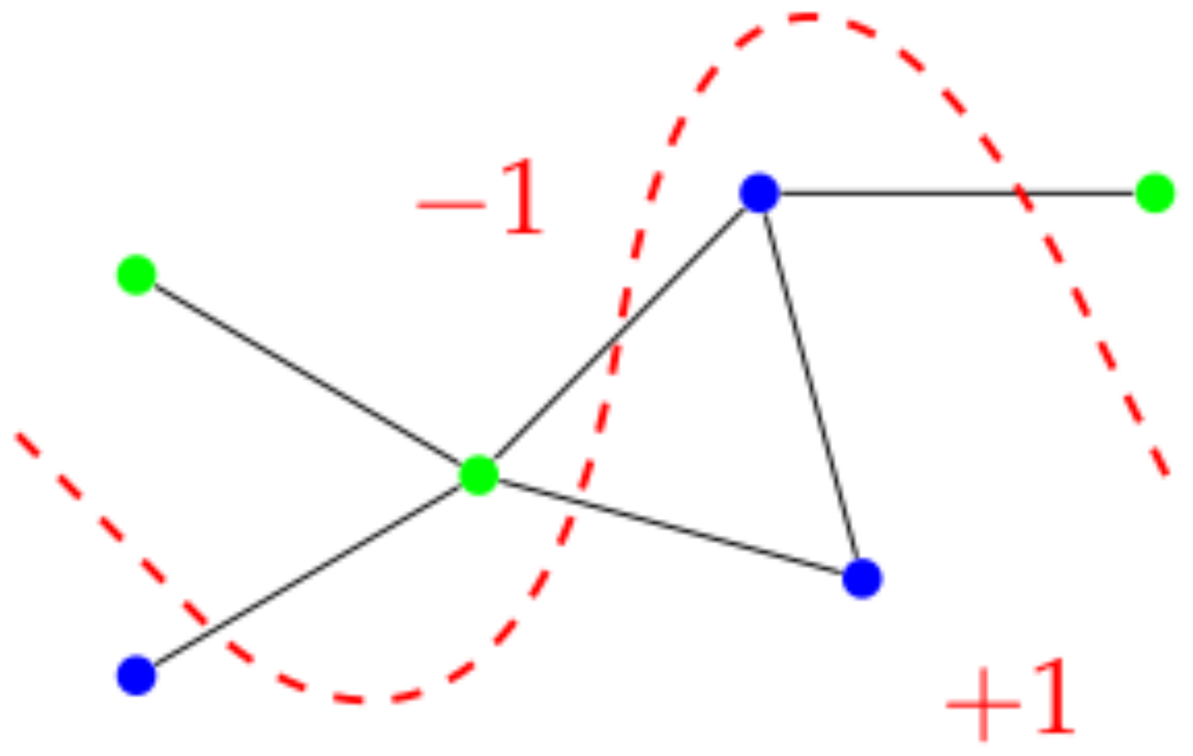
The applications of MagicSquare:

1. Device independent cryptography (Vazirani, Vidick 2014)
2. Verifying the result of a quantum computation (Reichardt, Unger, Vazirani,2012 , Mahadev 2018)
3. Delegation of quantum computation (Broadbent 2015)
4. Complexity theory: $\text{MIP}^* = \text{RE}$ (Ji, Natarajan, Vidick, Wright, Yuen 2020)
5. Physics (Bell's Theorem): Nature can generate correlations that would be impossible to generate based on classical mechanics (Bell 1964, Nobel Prize in Physics 2022)

MaxCut or Max-2-Colouring

Max-2-Colouring (MaxCut)

$G = (V, E)$

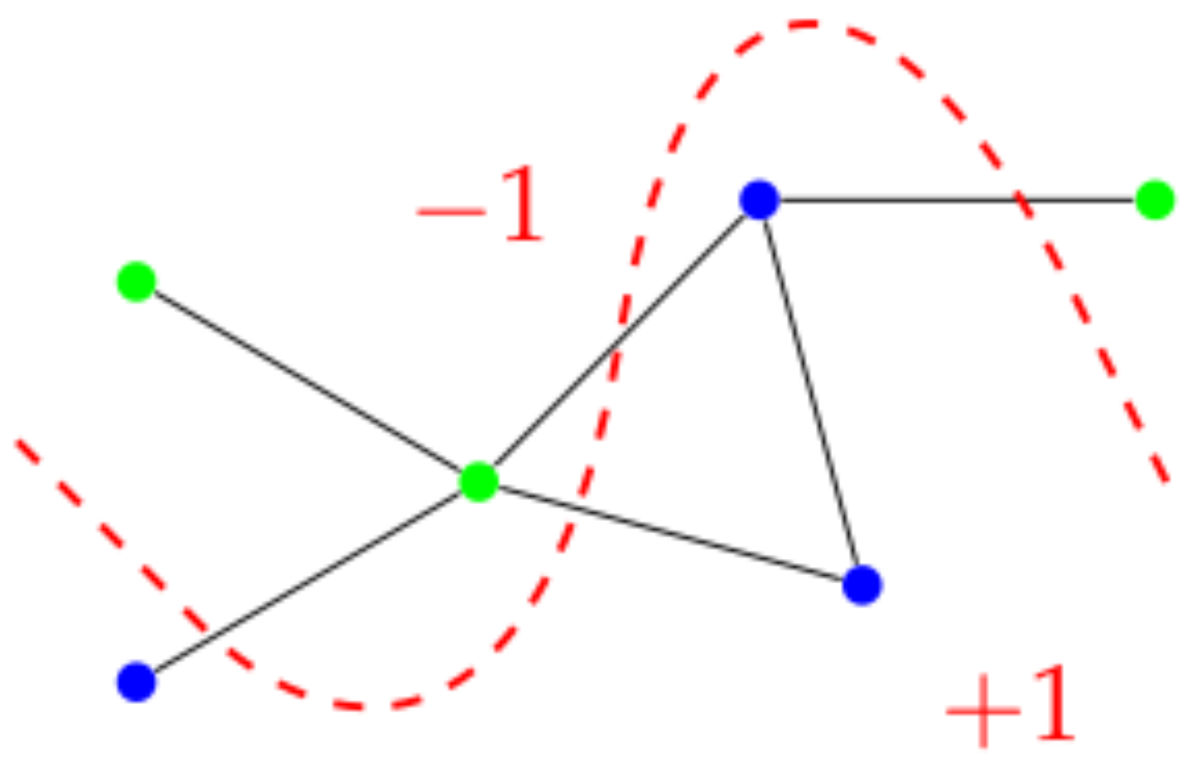


max $\sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$

s.t. x_i is a binary $\{-1, +1\}$ variable

MaxCut

$G = (V, E)$



$$\max \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

s.t. x_i is a binary $\{-1, +1\}$ variable

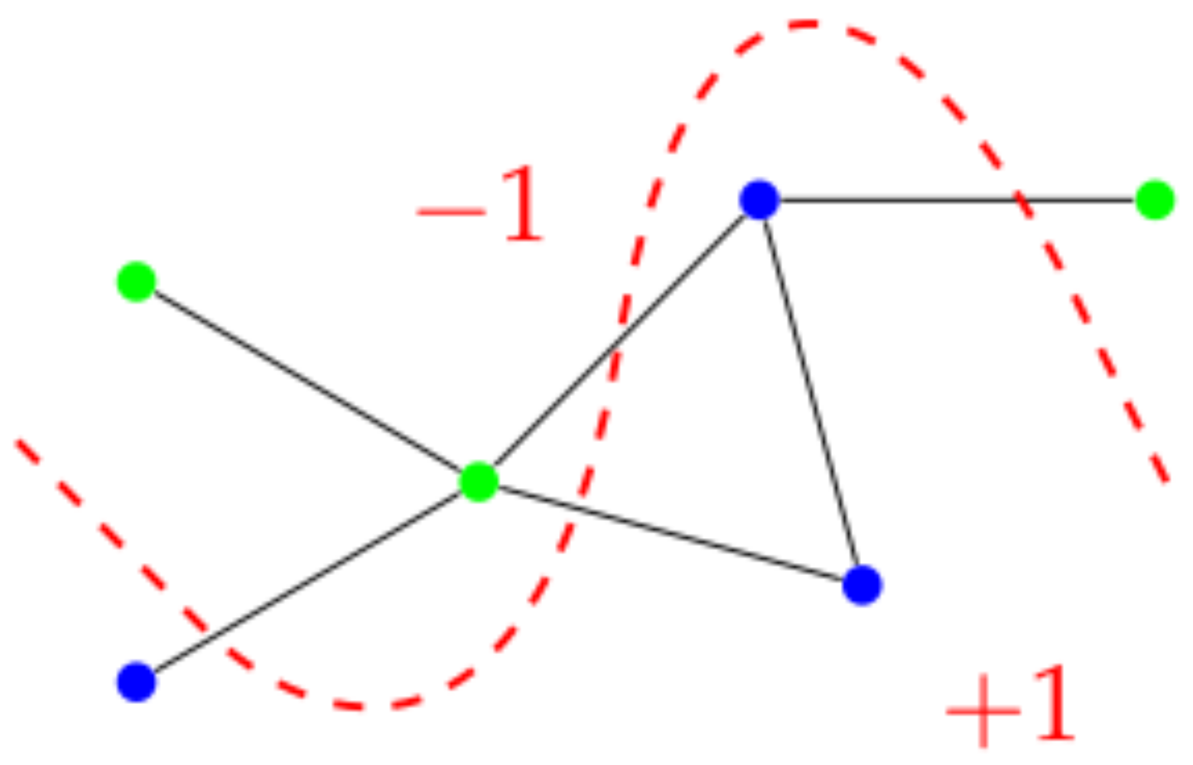
Noncommutative MaxCut

$$\max \sum_{(i,j)\in E} \frac{1 - \mathrm{tr}(X_i X_j)}{2}$$

s.t. X_i is a binary observable

MaxCut

$$G = (V, E)$$



$$\max \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

s.t. x_i is a binary $\{-1, +1\}$ variable

Noncommutative MaxCut

Recall: an observable is a unitary operator with $\{-1, +1\}$ eigenvalues.

Trace function tr is dimension-normalized.

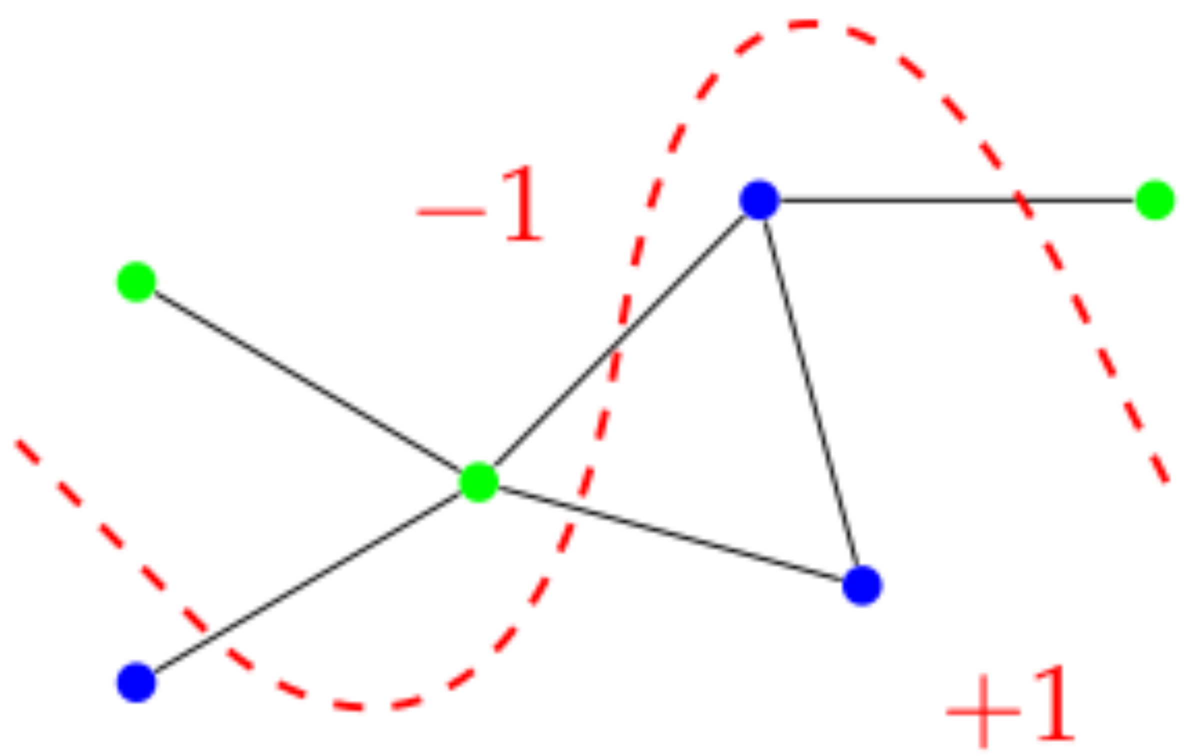
The optimization is over all finite dimensions.

$$\max \sum_{(i,j)\in E} \frac{1 - \mathrm{tr}(X_i X_j)}{2}$$

s.t. X_i is a binary observable

MaxCut (compact)

$G = (V, E)$, and let $\Gamma = [\gamma_{ij}]$ be the Laplacian



$$\max \sum_{(i,j)\in E} \gamma_{ij} x_i x_j$$

s.t. x_i is a binary $\{-1, +1\}$ variable

Noncommutative MaxCut

Recall: an observable is a unitary operator with $\{-1, +1\}$ eigenvalues.

Trace function tr is dimension-normalized.

The optimization is over all finite dimensions.

$$\max \operatorname{tr}\Big(\sum_{(i,j)\in E} \gamma_{ij} X_i X_j \Big)$$

s.t. X_i is a binary observable

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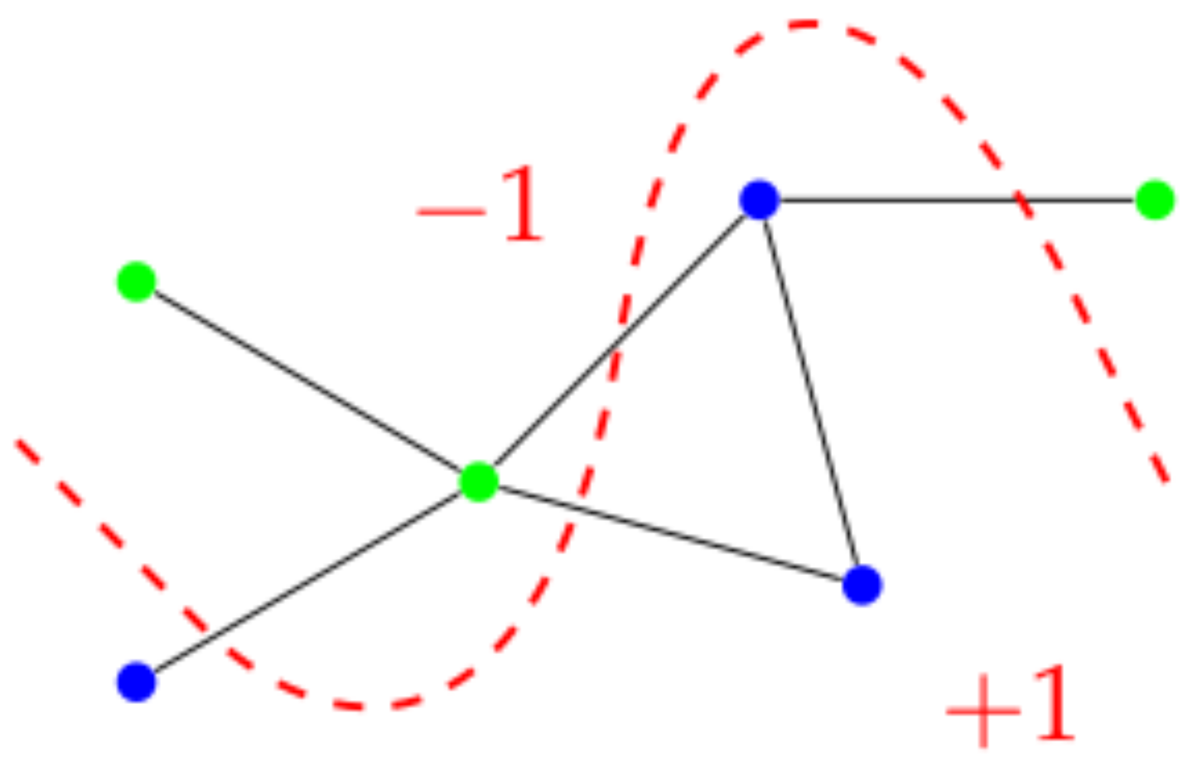


$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j$$

Noncommutative MaxCut

Obs(d) is the set of observables on a d-dimension vector space.

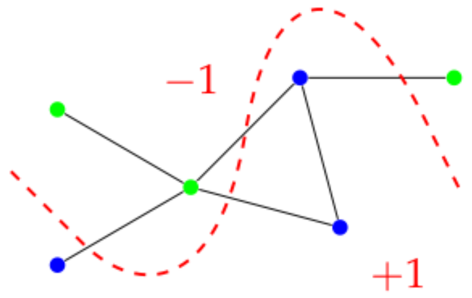
$\langle X_i, X_j \rangle = \mathrm{tr}(X_i^* X_j) = \mathrm{tr}(X_i X_j)$

$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

MaxCut: simple inequalities

$G = (V, E)$

$\Gamma = [\gamma_{ij}]$ is the Laplacian matrix of G



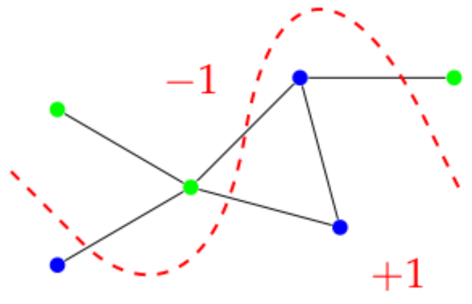
$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \leq \max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

Classical value | Noncommutative value

MaxCut: simple inequalities

$G = (V, E)$

$\Gamma = [\gamma_{ij}]$ is the Laplacian matrix of G



$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \quad \leq \quad \max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad \leq \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

Classical value

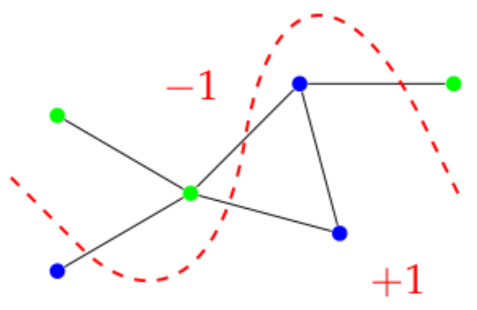
Noncommutative value

SDP value

MaxCut: simple inequalities

$G = (V, E)$

$\Gamma = [\gamma_{ij}]$ is the Laplacian matrix of G



$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \quad \leq \quad \max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad \leq \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

Classical value | Noncommutative value | SDP value

The reason we call the last column the SDP value is that

$$\max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$ can be restated as the semidefinite program $$\max_{\substack{V \geq 0 \\ \mathrm{diag}(V) = I}} \langle \Gamma, V \rangle$$

Do you recall that in noncommutative MagicSquare there were also some commutation relations?

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

X_{ij} are binary observables and satisfy the row and column commutation relations

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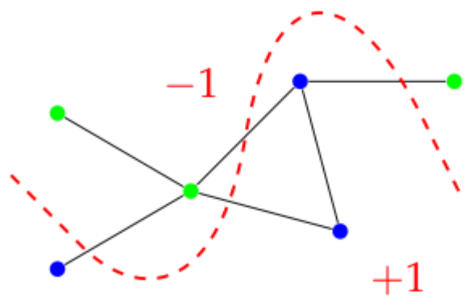
Why did not we impose these commutation relations in our NC-MaxCut?



$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

Noncommutative value

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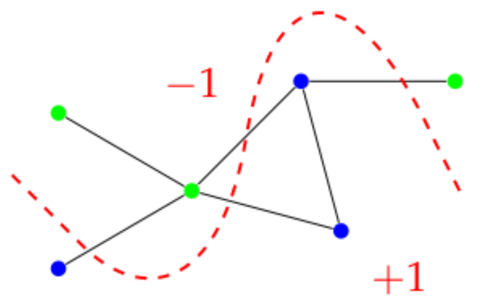


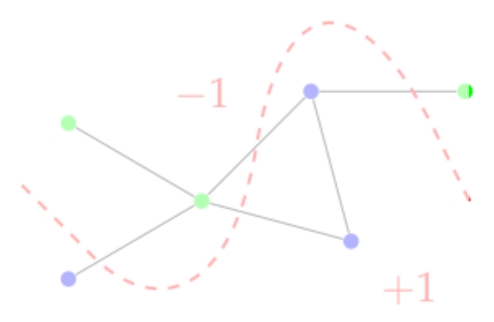
$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

Noncommutative value

Why did not we impose these commutation relations in our NC-MaxCut?

$G = (V, E)$

$\Gamma = [\gamma_{ij}]$ is the Laplacian matrix of G



$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

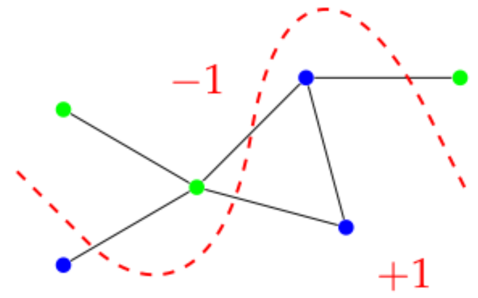
Noncommutative value

We can, but we obtain a different noncommutative generalization, we call Q-MaxCut:

$$\max_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j)\in E}} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

Quantum value

MaxCut: simple inequalities



$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \quad \le \quad \max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad \le \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

Classical value

Noncommutative value

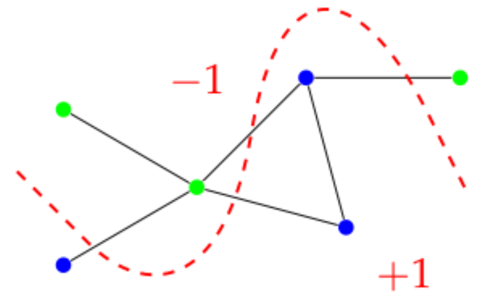
SDP value

$$\max_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j) \in E}} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$$

Quantum value

(Each of these values corresponds to a type of quantum strategy in the nonlocal games literature.)

MaxCut: simple inequalities

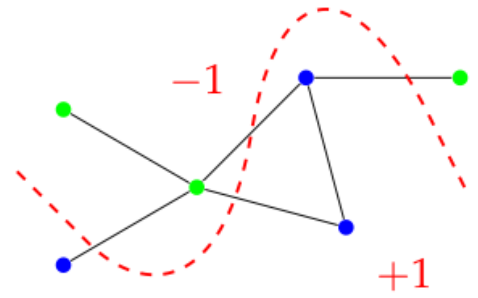


$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \quad \leq \quad \max_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j) \in E}} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad \leq \quad \max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad \leq \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

Classical value | Quantum value | Noncommutative value | SDP value

(Each of these values corresponds to a type of quantum strategy in the nonlocal games literature.)

MaxCut: all the flavours



MaxCut		**Q-MaxCut**		**NC-MaxCut**		**Vector-MaxCut**
$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j$	$\leq$	$\max_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j) \in E}} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$	$\leq$	$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$	$\leq$	$\max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$
Classical value		Quantum value		Noncommutative value		SDP value

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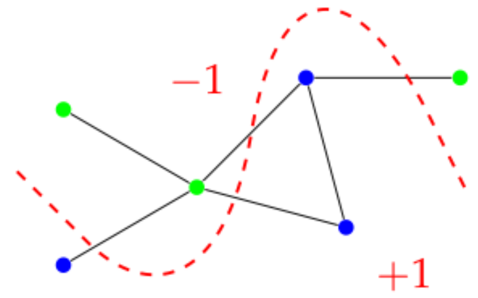
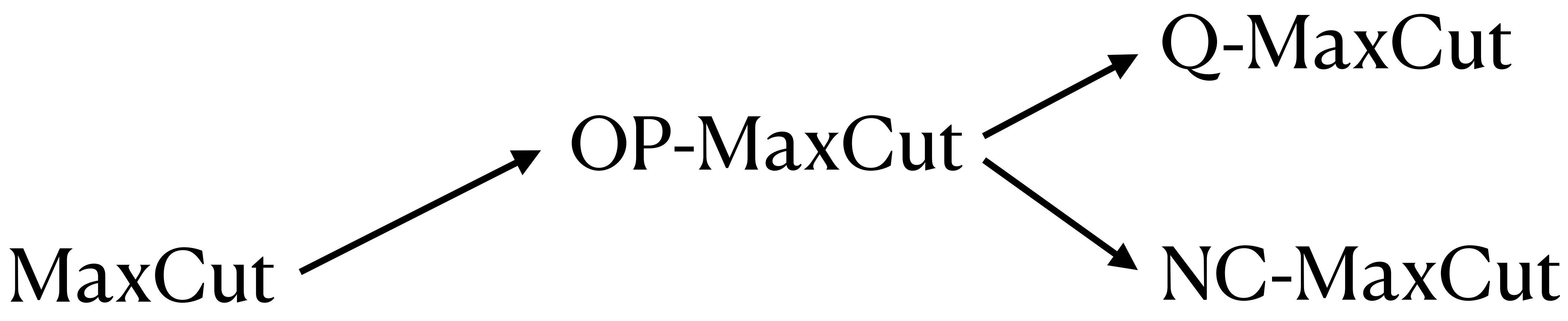


MaxCut		**Q-MaxCut**		**NC-MaxCut**		**Vector-MaxCut**
$\max\limits_{x_i \in \{\pm 1\}} \sum\limits_{(i,j)\in E} \gamma_{ij} x_i x_j$	$\leq$	$\max\limits_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j) \in E}} \sum\limits_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$	$\leq$	$\max\limits_{X_i \in \mathrm{Obs}(d)} \sum\limits_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$	$\leq$	$\max\limits_{\substack{v_i \in \mathbb{R}^d \\ \lVert v_i \rVert = 1}} \sum\limits_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$
Classical value		Quantum value		Noncommutative value		SDP value



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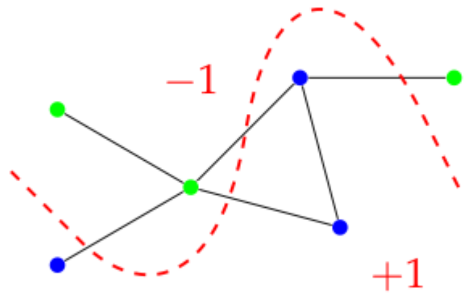
MaxCut: best algorithms



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Classical value | Quantum value | Noncommutative value | SDP value

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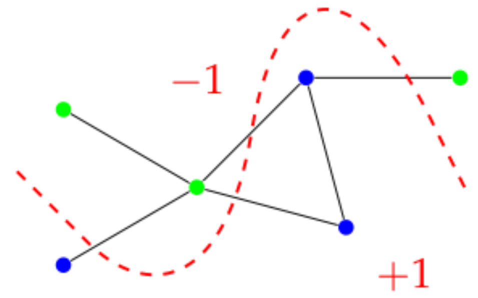

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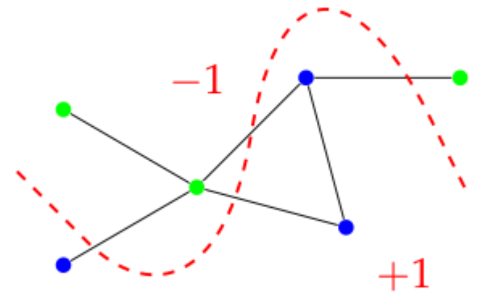
Classical value	Quantum value	Noncommutative value	SDP value
NP-hard (Karp)	undecidable?	undecidable?	polynomial-time

(Each of these values corresponds to a type of quantum strategy in the nonlocal games literature.)

MaxCut: Tsirelson's Theorem



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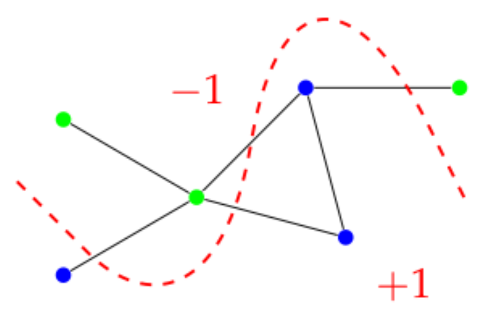
Classical value	Quantum value	Noncommutative value	SDP value
NP-hard (Karp)	undecidable?	polynomial-time (Tsirelson)	polynomial-time

MaxCut: Tsirelson's Proof



$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad = \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

Noncommutative value

SDP value

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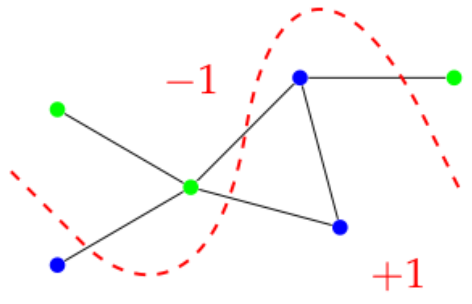

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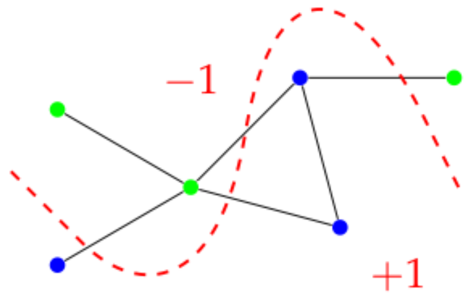
Noncommutative value | SDP value

There exists an isometry $v \mapsto X$ such that when v is a unit vector, X is a binary observable.

MaxCut: Tsirelson's Proof



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Noncommutative value — SDP value

There exists an isometry $v \mapsto X$ such that when v is a unit vector, X is a binary observable.

Apply the isometry to the vectors in the SDP solution

$$v_1 \mapsto X_1$$

$$v_2 \mapsto X_2$$

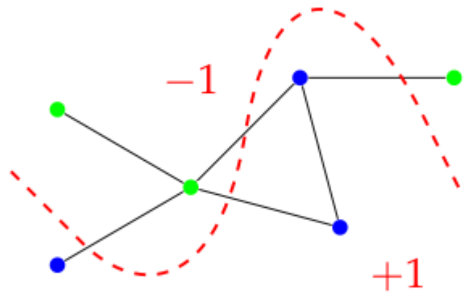
$$\vdots$$

$$v_n \mapsto X_n.$$

Now $X_1, \ldots, X_n$ is a feasible solution in NC-Max-Cut.

And it has the same objective value as the SDP solution.

MaxCut: Tsirelson's Proof



$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad = \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

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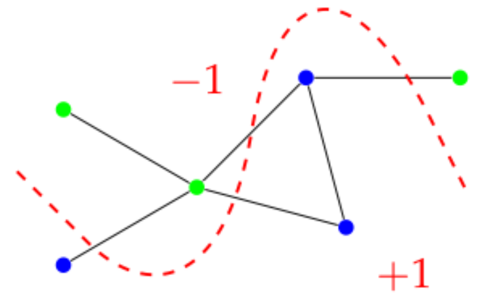
Construction of Tsirelson's isometry

MaxCut: Tsirelson's Proof



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Noncommutative value

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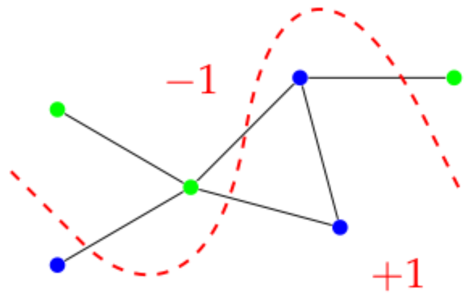
Construction of Tsirelson's isometry

Let $\sigma_1, \ldots, \sigma_d$ be the Weyl-Brauer operators:

They are binary observables, and they pairwise anticommute.

MaxCut: Tsirelson's Proof



$$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij}\langle X_i, X_j\rangle \quad = \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij}\langle v_i, v_j\rangle$$

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Construction of Tsirelson's isometry

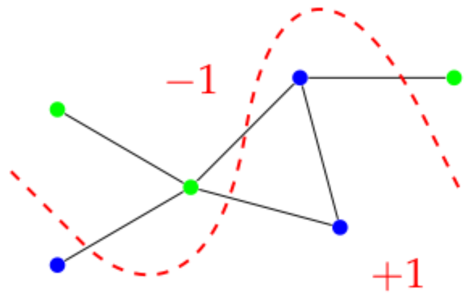
Let $\sigma_1, \ldots, \sigma_d$ be the Weyl-Brauer operators:

They are binary observables, and they pairwise anticommute.

Then the isometry on $v = (a_1, \ldots, a_d) \in \mathbb{R}^d$ is given by

$$v \mapsto a_1\sigma_1 + \cdots + a_d\sigma_d$$

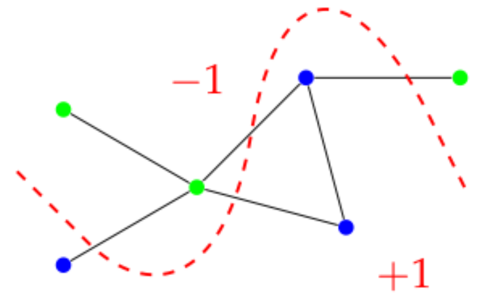
MaxCut: Best Algorithms



$$\max_{x_i\in\{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \;\le\; \max_{\substack{X_i\in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j)\in E}} \sum_{(i,j)\in E} \gamma_{ij}\langle X_i, X_j\rangle \;\le\; \max_{X_i\in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij}\langle X_i, X_j\rangle \;=\; \max_{\substack{v_i\in\mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij}\langle v_i, v_j\rangle$$

Classical value	Quantum value	Noncommutative value	SDP value
NP-hard (Karp)	undecidable?	polynomial-time (Tsirelson)	polynomial-time

MaxCut: Best Algorithms



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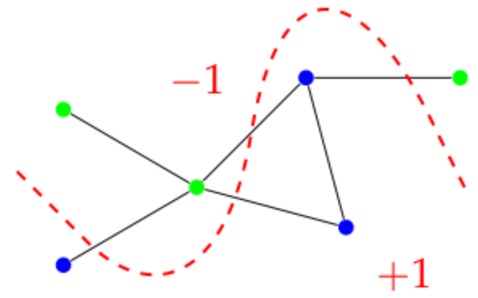
MaxCut: Best Algorithms



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Classical value	**Quantum value**	Noncommutative value	SDP value
NP-hard (Karp)	**undecidable?**	polynomial-time (Tsirelson)	polynomial-time
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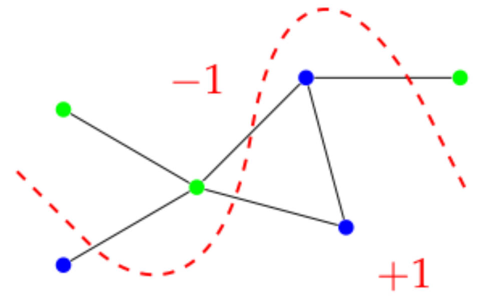
MaxCut: Best Algorithms



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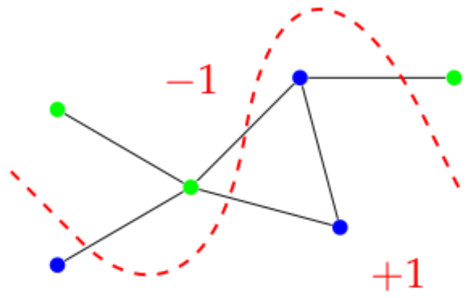
Classical value	**Quantum value**	Noncommutative value	SDP value
NP-hard (Karp)	**undecidable?**	polynomial-time (Tsirelson)	polynomial-time
		$v \mapsto a_1\sigma_1 + \cdots + a_d\sigma_d$	$v = (a_1, \ldots, a_d) \in \mathbb{R}^d$

Tsirelson's isometry produces highly noncommutative operators.
So we cannot use it for the quantum value.

MaxCut: Best Algorithms



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Classical value	Quantum value	Noncommutative value	SDP value
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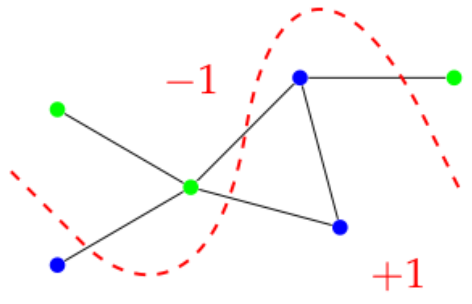
MaxCut: Best Algorithms



$$\max_{x_i\in\{\pm1\}} \sum_{(i,j)\in E} \gamma_{ij}x_ix_j \;\le\; \max_{\substack{X_i\in\mathrm{Obs}(d)\\ [X_i,X_j]=I\\ \text{for all } (i,j)\in E}} \sum_{(i,j)\in E} \gamma_{ij}\langle X_i, X_j\rangle \;\le\; \max_{X_i\in\mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij}\langle X_i, X_j\rangle \;=\; \max_{\substack{v_i\in\mathbb{R}^d\\ \|v_i\|=1}} \sum_{(i,j)\in E} \gamma_{ij}\langle v_i, v_j\rangle$$

Classical value	Quantum value	Noncommutative value	SDP value
0.878-approximation	undecidable?	polynomial-time (Tsirelson)	polynomial-time
(Goemans and Williamson)		$v \mapsto a_1\sigma_1 + \cdots + a_d\sigma_d$	$v = (a_1, \dots, a_d) \in \mathbb{R}^d$

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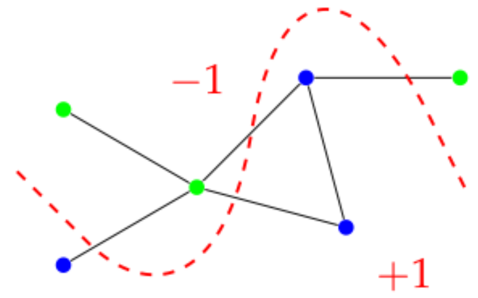

$$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j \quad \leq \quad \max_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j) \in E}} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad \leq \quad \max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle \quad = \quad \max_{\substack{v_i \in \mathbb{R}^d \\ \|v_i\| = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$$

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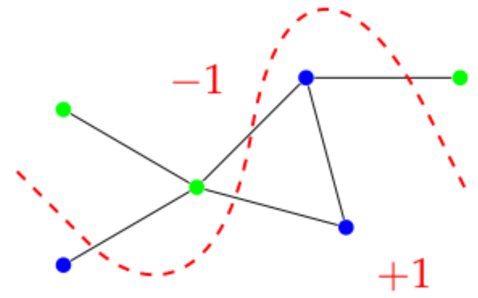

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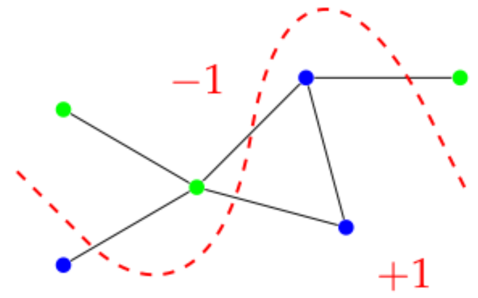

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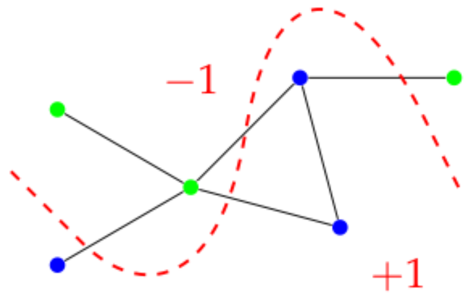
If we apply this to SDP vectors, it does not yield ± 1.

MaxCut: Best Algorithms



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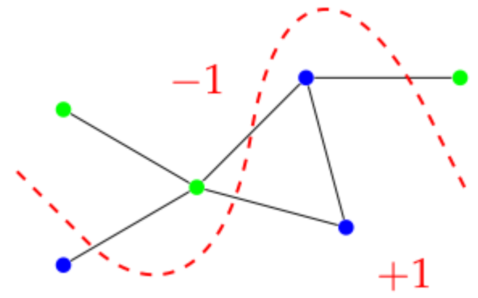
So we need to use a **rounding scheme**.

MaxCut: Best Algorithms



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1. Sample a unit vector $r \in \mathbb{R}^d$
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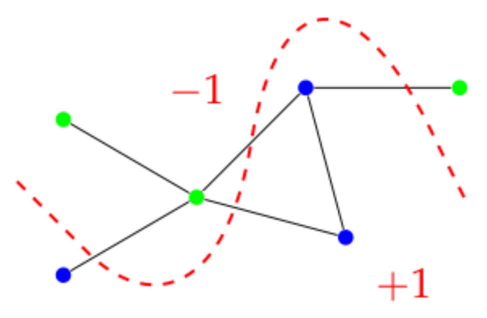
$$v = (a_1, \ldots, a_d) \mapsto \mathrm{sign}(a_1 r_1 + \cdots + a_d r_d)$$

MaxCut: Best Algorithms



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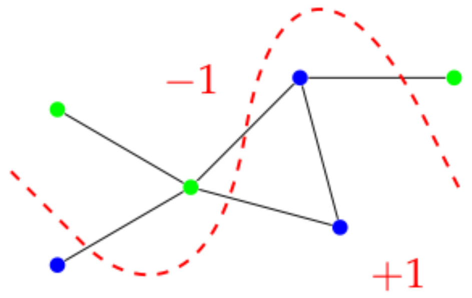
That is why we loose a little and get a 0.878-approximation.

MaxCut: Best Algorithms



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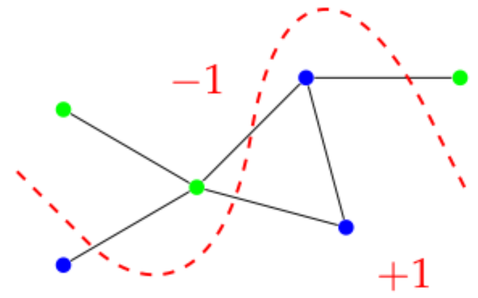

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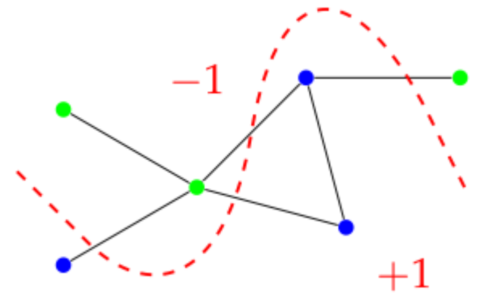
MaxCut: Hardness



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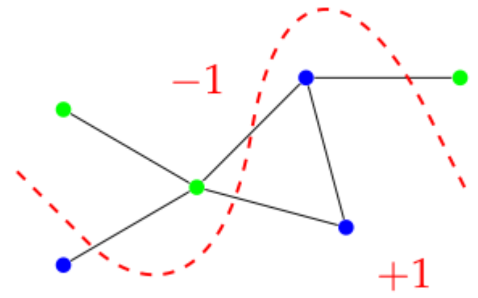

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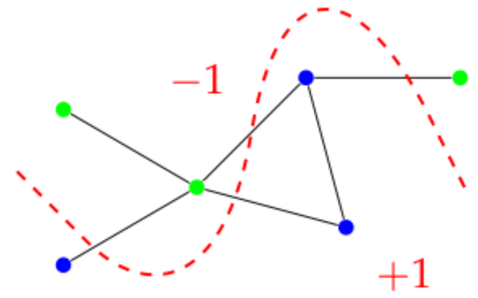

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Assuming unique games conjecture, this is the best efficient algorithm.

(Khot, Kindler, Mossel, O'Donnell)

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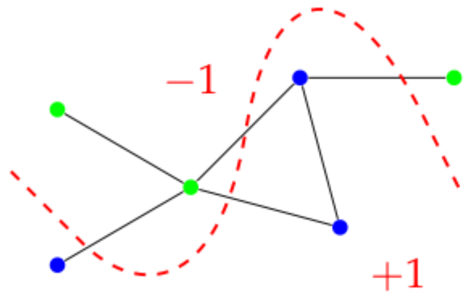

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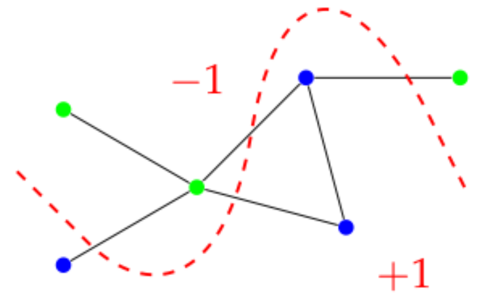

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Classical value | Quantum value | Noncommutative value | SDP value

Classical value: 0.878-approximation

(Goemans and Williamson)

$v \mapsto a_1 r_1 + \cdots + a_d r_d$

Quantum value: Assuming **quantum unique games conjecture**, Goemans-Williamson is the best efficient algorithm.

(M., Spirig)

Noncommutative value: polynomial-time (Tsirelson)

$v \mapsto a_1 \sigma_1 + \cdots + a_d \sigma_d$

SDP value: polynomial-time

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Assuming unique games conjecture, this is the best efficient algorithm.

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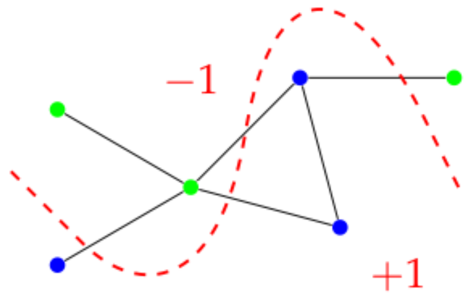

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0.878-approximation

(Goemans and Williamson)

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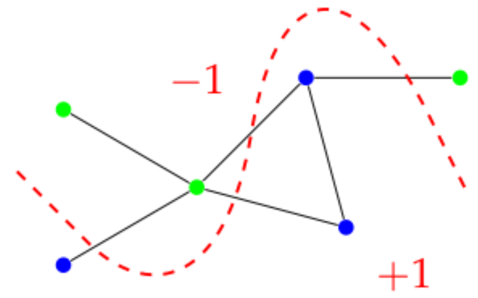
MaxCut: Complexity transition diagrams



MaxCut		**Q-MaxCut**		**NC-MaxCut**		**SDP-MaxCut**
$\max_{x_i \in \{\pm 1\}} \sum_{(i,j)\in E} \gamma_{ij} x_i x_j$	$\leq$	$\max_{\substack{X_i \in \mathrm{Obs}(d) \\ [X_i, X_j] = I \\ \text{for all } (i,j)\in E}} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$	$\leq$	$\max_{X_i \in \mathrm{Obs}(d)} \sum_{(i,j)\in E} \gamma_{ij} \langle X_i, X_j \rangle$	$=$	$\max_{\substack{v_i \in \mathbb{R}^d \\ \lVert v_i \rVert = 1}} \sum_{(i,j)\in E} \gamma_{ij} \langle v_i, v_j \rangle$
0.878-approximation		0.878-approximation		polynomial-time		polynomial-time

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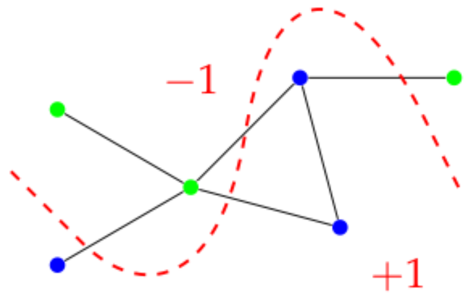

MaxCut		Q-MaxCut		NC-MaxCut		SDP-MaxCut
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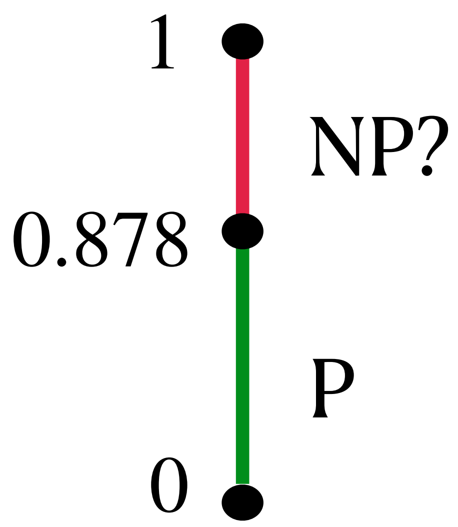
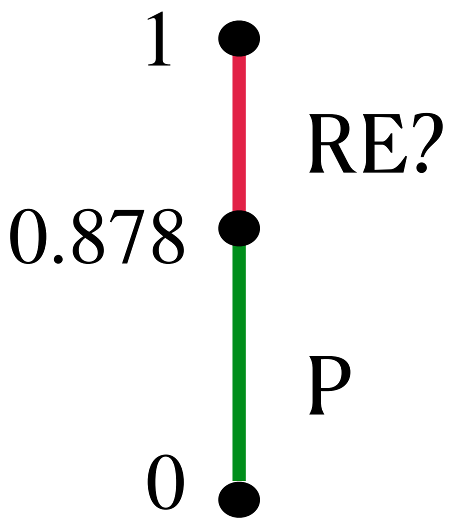
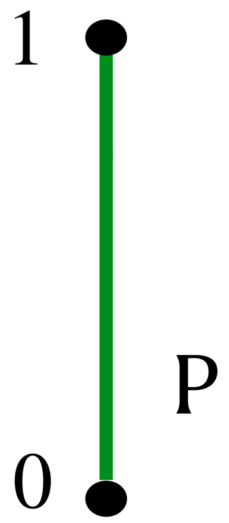
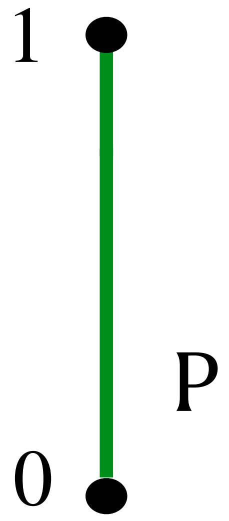
0.878-approximation		0.878-approximation		polynomial-time		polynomial-time









RE is short for recursively enumerable. Being RE-hard is synonymous with undecidable.

Types of PCPs and UGCs

- We have **PCP**, **Q-PCP**, and **NC-PCP**:
  - **PCP** says **LabelCover** is hard to approximate (Arora, Safra, Lund, Motwani, Sudan, Szegedy, Raz, Håstad).
  - **Q-PCP** says **Q-LabelCover** is hard to approximate (Ji, Natarajan, Vidick, Wright, Yuen).
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- We cannot have **NC-UGC.** This is because there is a good algorithm for **NC-UniqueLabelCover** (Kempe, Regev, Toner).
- But **UGC** and **Q-UGC** are still in the realm of possibilities.

Classical CSPs (commutation) → Q-CSPs (some commutation)

has the same theory of approximation

Q-CSP is short for Quantum-CSP.

Classical CSPs (commutation) ⟶ Q-CSPs (some commutation)

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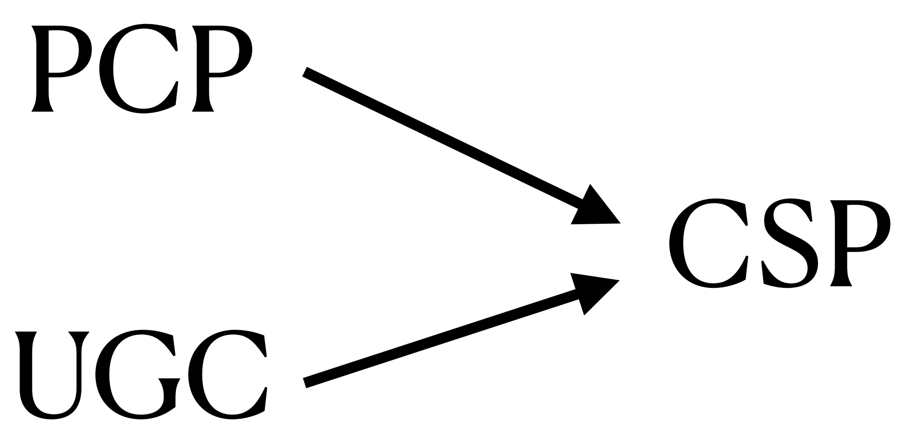

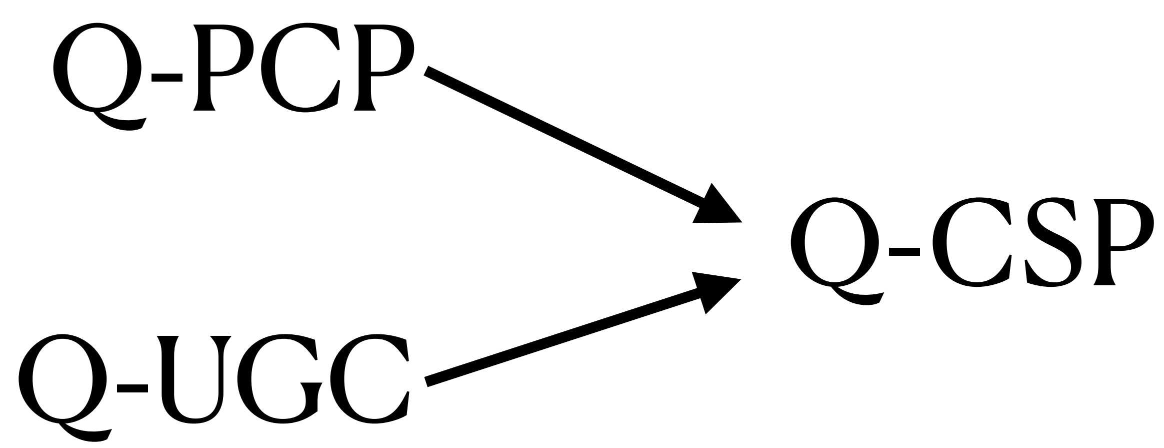

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Classical CSPs (commutation) → Q-CSPs (some commutation)

has the same theory of approximation

Exists only for 2-CSPs

Classical CSPs (commutation) → NC-CSPs (no commutation)

exiting twists!

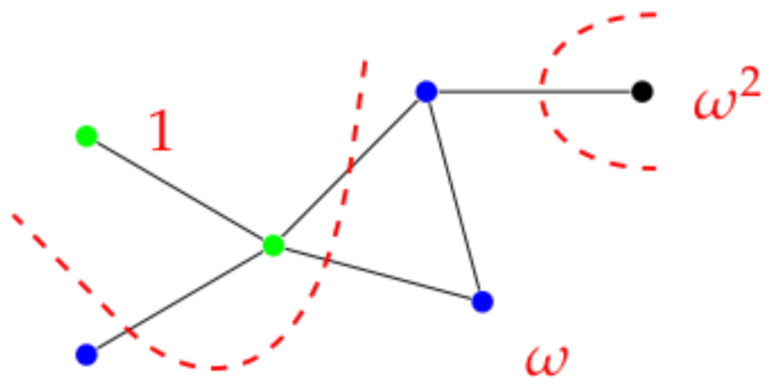
but similar proof techniques will perhaps work?

Q-CSP is short for Quantum-CSP.

NC-CSP is short for Noncommutative-CSP.

In a 2-CSP every constraint involves only two variables.

Another example: Max-3-Cut (or Max-3-Colouring)



Max-3-Cut $\leq$ **NC-Max-3-Cut** $\leq$ **SDP-Max-3-Cut**

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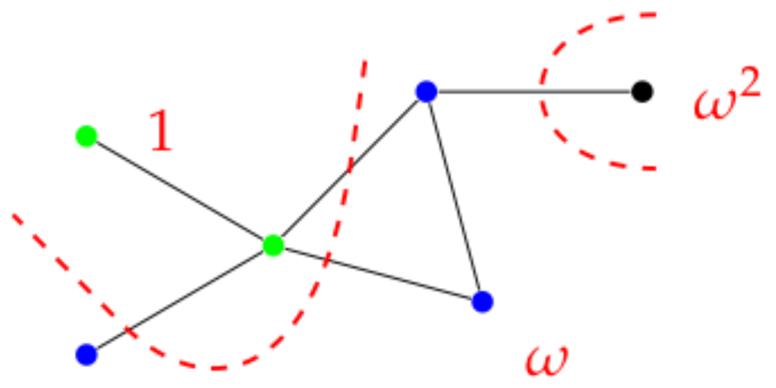

Unlike NC-MaxCut (which can be solved in poly-time), we know NC-Max-3-Cut is undecidable (Ji)

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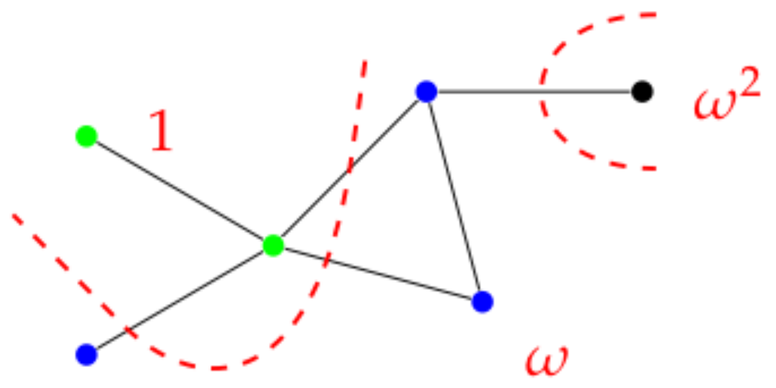

Max-3-Cut	$\leq$	NC-Max-3-Cut	$\leq$	SDP-Max-3-Cut
0.836-approximation Algorithm: Frieze and Jerrum Goemans and Williamson de Klerk, Pasechnik, and Warners		0.864-approximation Algorithm: Culf, M., Spirig		

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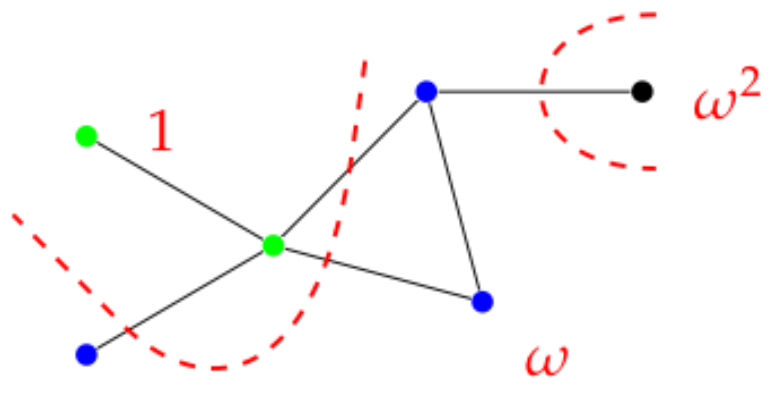

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Max-3-Cut

0.836-approximation

Algorithm: Frieze and Jerrum
Goemans and Williamson
de Klerk, Pasechnik, and Warners

Hardness: Khot, Kindler, Mossel, O'Donnell

NC-Max-3-Cut

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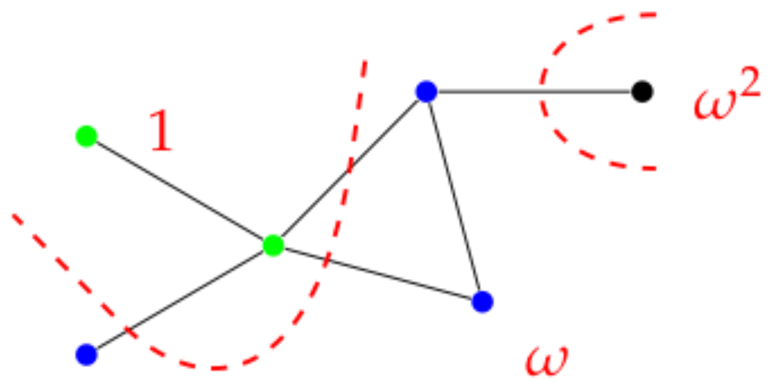

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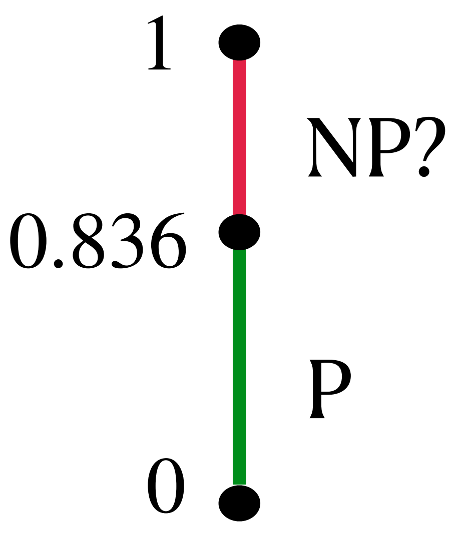

NC-Max-3-Cut

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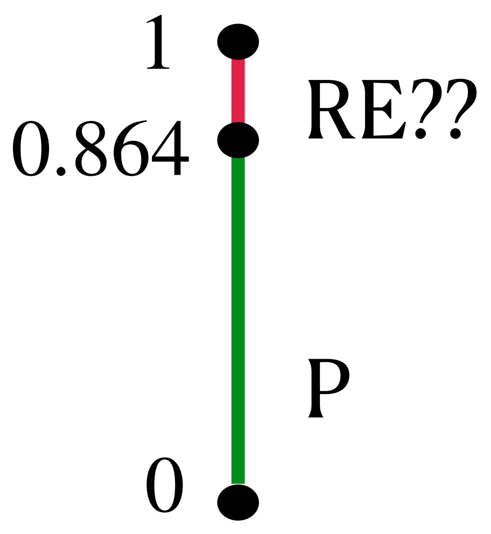
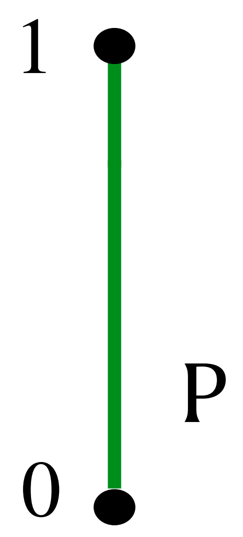


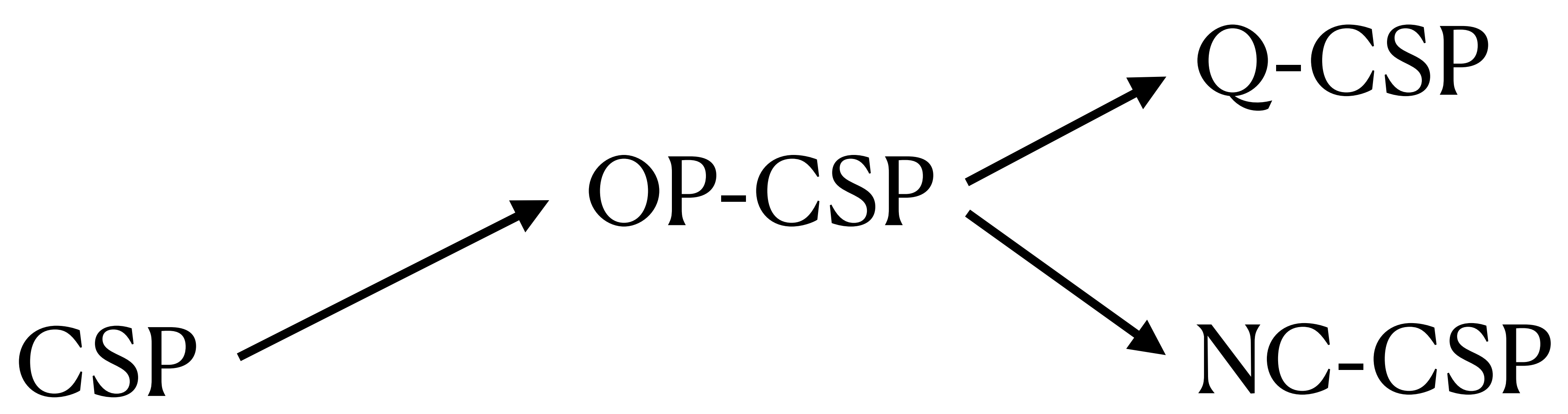
SDP-Max-3-Cut



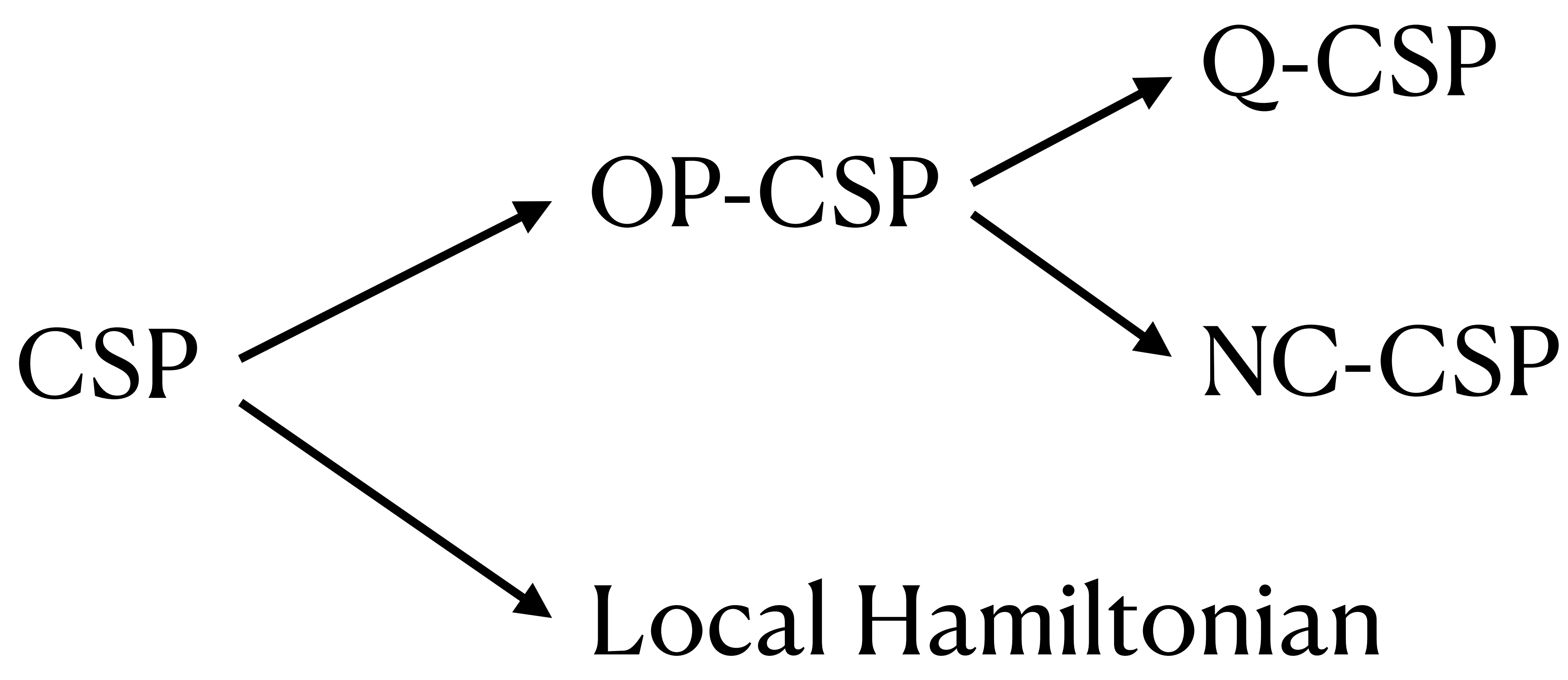
Where to take this next (final part)

NC-CSPs and quantum complexity classes

Quantum generalizations of CSPs



Quantum generalizations of CSPs



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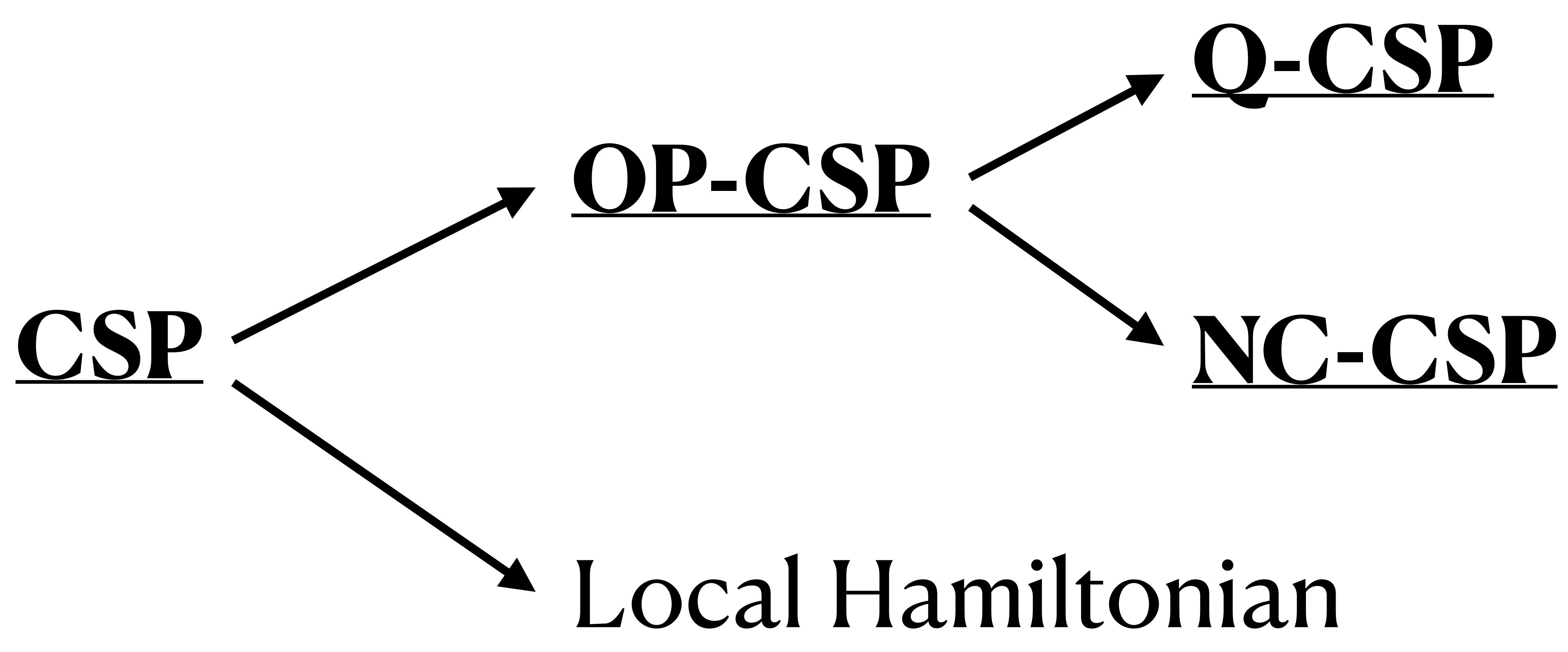
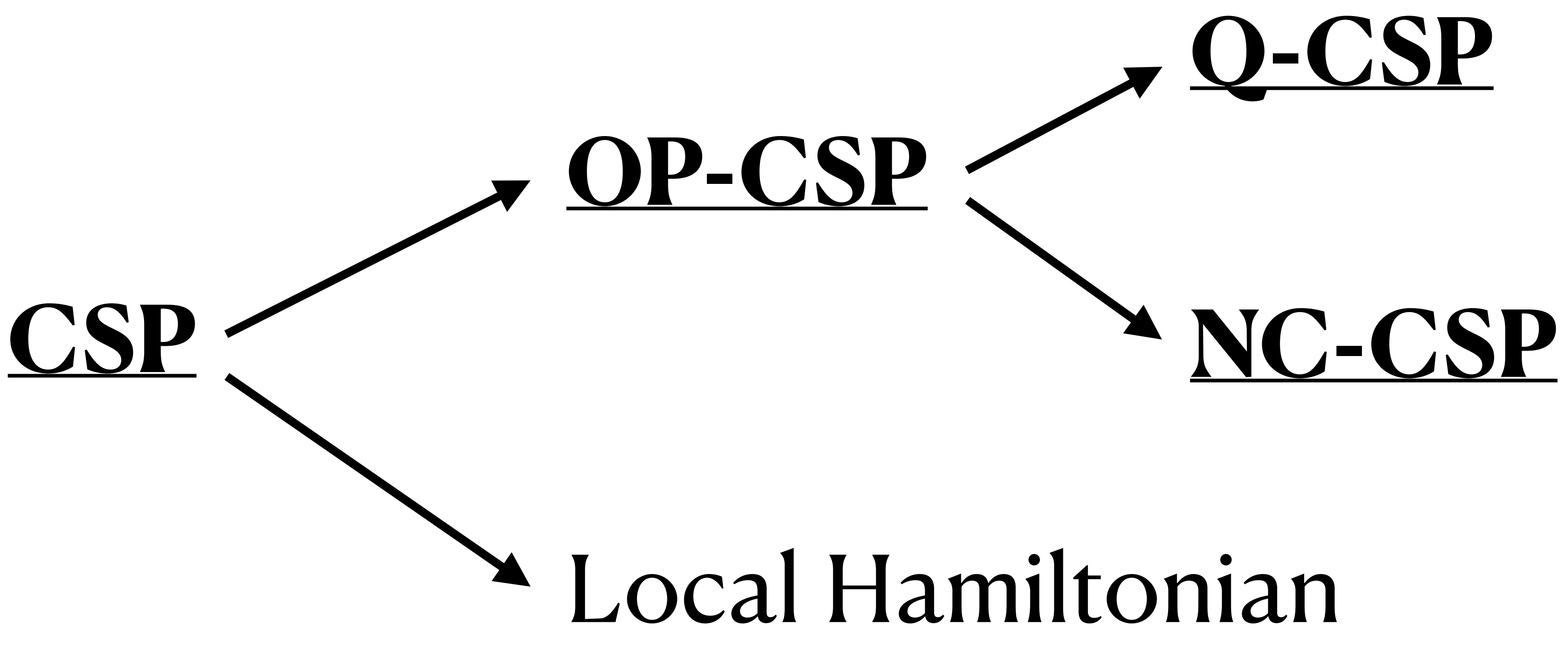


We know much more about the hardness of approximation of the upper branch.

Quantum generalizations of CSPs



We know much more about the hardness of approximation of the upper branch.

For example we have a PCP theorem for every member. No PCP for Local Hamiltonian though!

Why this difference between OP-CSP and Local Hamiltonian?

The algebraic nature of CS tools (sum-check protocol, low-degree testing, Fourier analysis on the hypercube)

fits

the algebraic nature of CSPs and OP-CSPs

CSPs: commutative algebras

NC-CSPs: matrix algebras

Local Hamiltonians: not algebraic

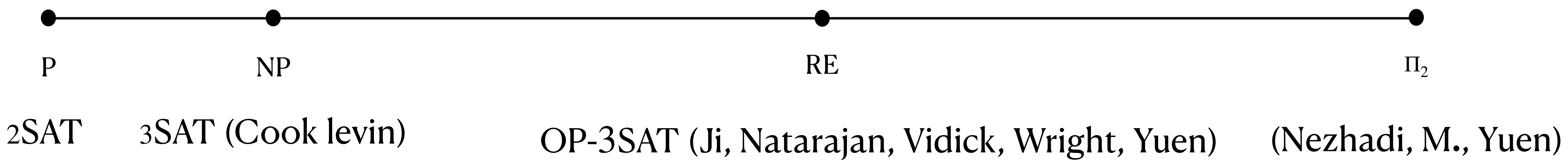
OP-CSPs are expressive

2SAT 3SAT (Cook levin) OP-3SAT (Ji, Natarajan, Vidick, Wright, Yuen)

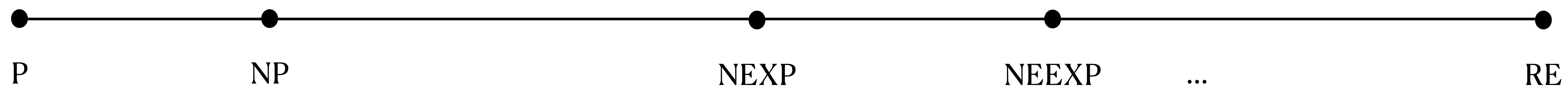
OP-CSPs are expressive

P	NP	RE	Π_2
2SAT	3SAT (Cook levin)	OP-3SAT (Ji, Natarajan, Vidick, Wright, Yuen)	(Nezhadi, M., Yuen)

OP-CSPs are expressive



They also capture all the nondeterministic classes



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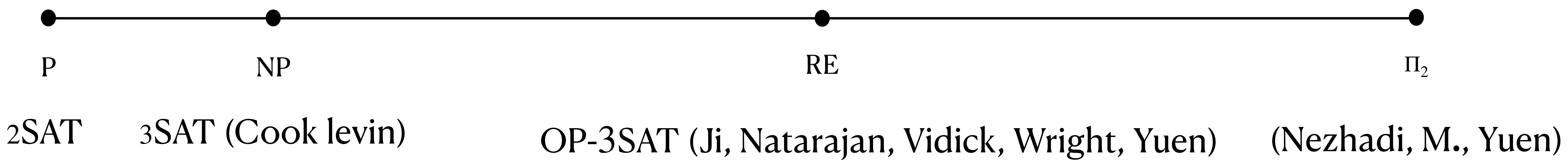

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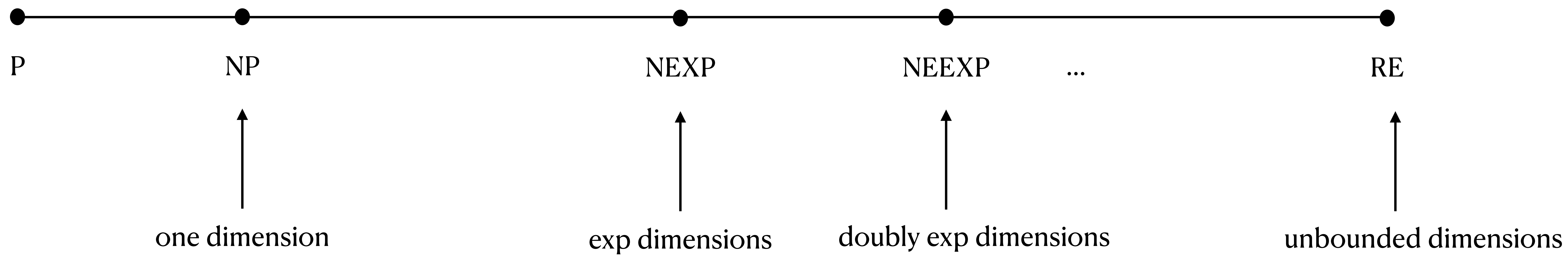
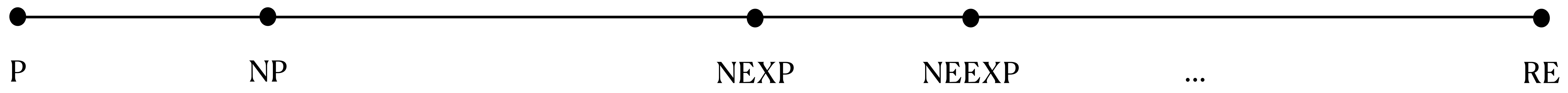


But they skip on quantum complexity classes



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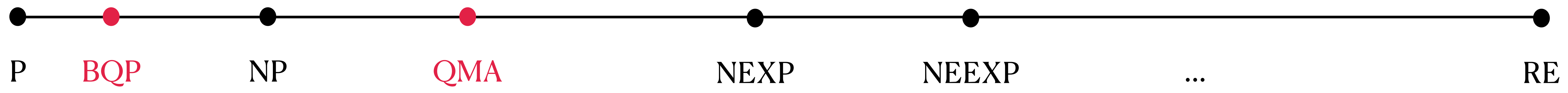
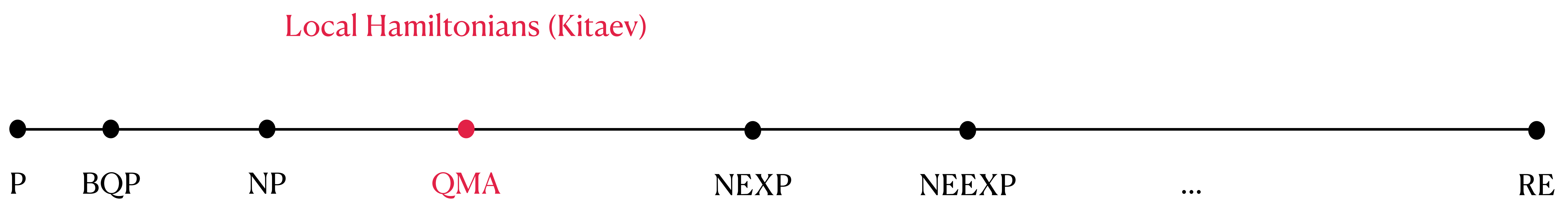
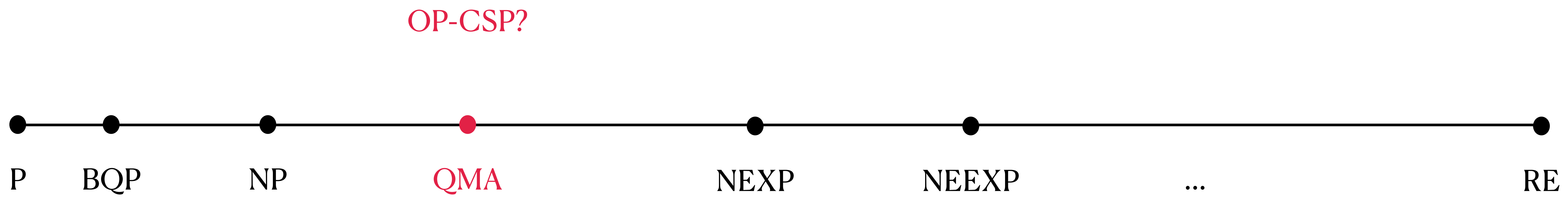


Local-Hamiltonian fills the gap



Open problem



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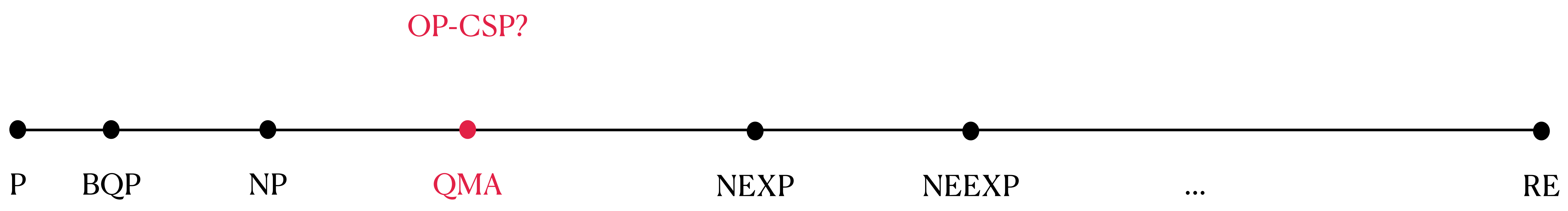
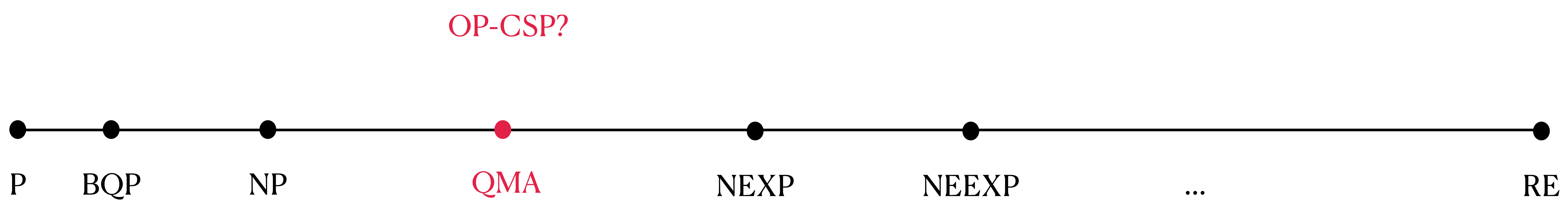


- Restricting the dimension of observable => nondeterministic classes
- Requiring that the observables are efficiently implementable

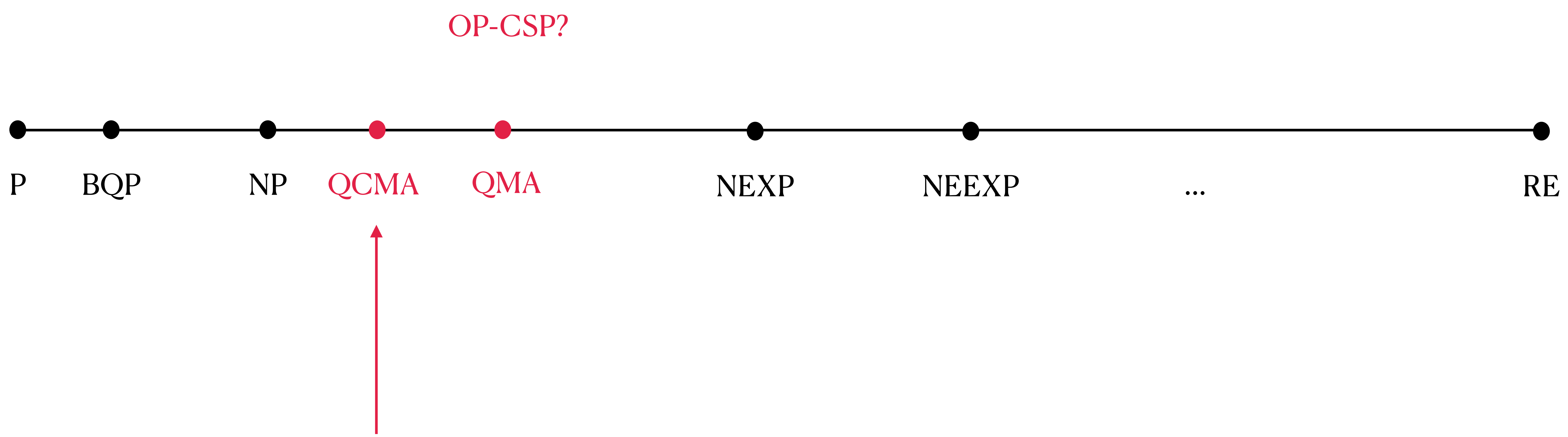
Open problem



$$\max \sum tr(X_i X_j)$$

s.t. X_i is an observable with an efficient circuit

Open problem



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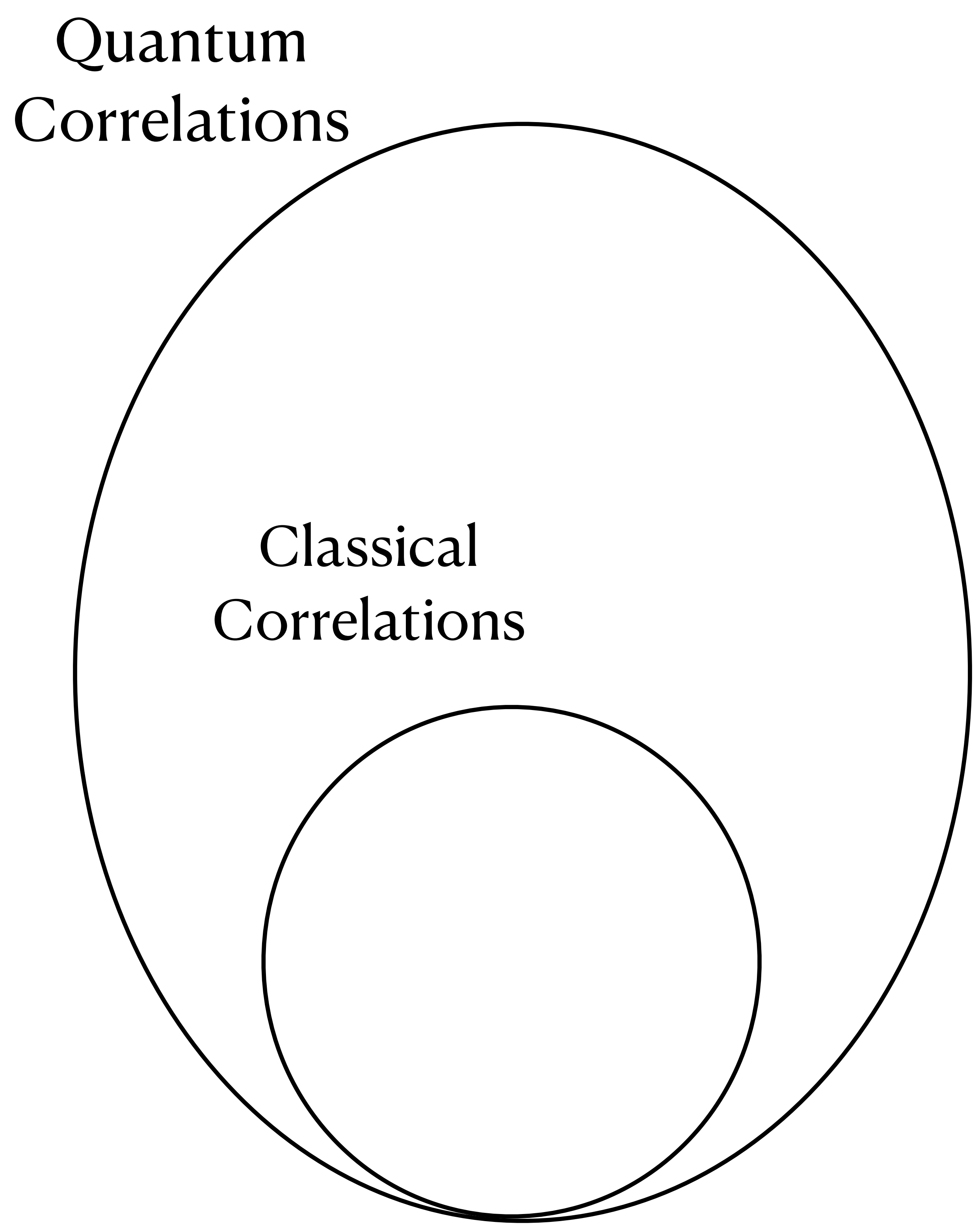
Set of correlations



Efficiently generated correlations: correlations that are realizable