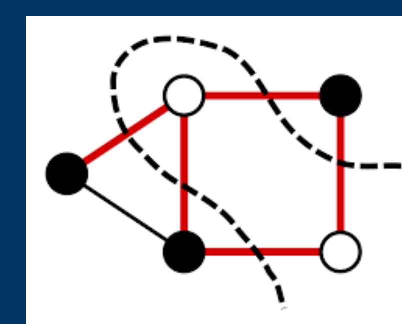
Noncommutativity, CSPs, and Quantum Computation

Maximize $< \phi | A_0 B_0 + A_0 B_1 + A_1 B_0 - A_1 B_1 | \phi >$



Max-Cut

Hamoon Mousavi (Simons at Berkeley)

Plan

- Noncommutative CSPs: MIP* and nonlocal games
- Quantum CSPs: local-Hamiltonian problems
    - Each captures an important physical/quantum info concept
    - Computational aspects
- Core message is an open problem: Why this divide in quantum?
    - No divide in the classical CS between the two concepts:
        - Proof verification
        - One round multiplayer games

Point 1:

The algebraic nature of alphabets

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The algebraic nature of alphabets

Transcript of a Turing machine:

0	1	0	1
0	0	0	1
1	0	0	1

Point 1:

The algebraic nature of alphabets

The transcript follows local rules: for example x_{32} is the AND of x_{22} and x_{23}

x_{11}	x_{12}	x_{13}	x_{14}
x_{21}	x_{22}	x_{23}	x_{24}
x_{31}	x_{32}	x_{33}	x_{34}

3SAT formula:

$$(\sim x_{32} \vee x_{22}) \wedge (\sim x_{32} \vee x_{23}) \wedge (x_{32} \vee \sim x_{22} \vee \sim x_{23})$$

Point 1:

The algebraic nature of alphabets

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3SAT formula:

$$(\sim x_{32} \vee x_{22}) \wedge (\sim x_{32} \vee x_{23}) \wedge (x_{32} \vee \sim x_{22} \vee \sim x_{23})$$

- Boolean algebra
- $\mathbb{F}_p^n$
- Boolean hypercube
- etc.

ON COMPUTABLE NUMBERS, WITH AN APPLICATION TO THE ENTSCHEIDUNGSPROBLEM

By A. M. Turing.

[Received 28 May, 1936.—Read 12 November, 1936.]

Computing is normally done by writing certain symbols on paper. We may suppose this paper is divided into squares like a child's arithmetic book. In elementary arithmetic the two-dimensional character of the paper is sometimes used. But such a use is always avoidable, and I think that it will be agreed that the two-dimensional character of paper is no essential of computation. I assume then that the computation is carried out on one-dimensional paper, *i.e.* on a tape divided into squares. I shall also suppose that the number of symbols which may be printed is finite. If we were to allow an infinity of symbols, then there would be symbols differing to an arbitrarily small extent†. The effect of this restriction of the number of symbols is not very serious. It is always possible to use sequences of symbols in the place of single symbols. Thus an Arabic numeral such as 17 or 999999999999999 is normally treated as a single symbol. Similarly in any European language words are treated as single symbols (Chinese, however, attempts to have an enumerable infinity of symbols). The differences from our point of view between the single and compound symbols is that the compound symbols, if they are too lengthy, cannot be observed at one glance. This is in accordance with experience. We cannot tell at a glance whether 9999999999999999 and 999999999999999 are the same.

The behaviour of the computer at any moment is determined by the symbols which he is observing, and his "state of mind" at that moment. We may suppose that there is a bound B to the number of symbols or squares which the computer can observe at one moment. If he wishes to

† If we regard a symbol as literally printed on a square we may suppose that the square is $0 \leqslant x \leqslant 1$, $0 \leqslant y \leqslant 1$. The symbol is defined as a set of points in this square, viz. the set occupied by printer's ink. If these sets are restricted to be measurable, we can define the "distance" between two symbols as the cost of transforming one symbol into the other if the cost of moving unit area of printer's ink unit distance is unity, and there is an [illegible]

Noncommutative CSPs

by means of examples

Magic Square

$x_{ij} \in \{+1, -1\}$

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

What should be the noncommutative alphabet?

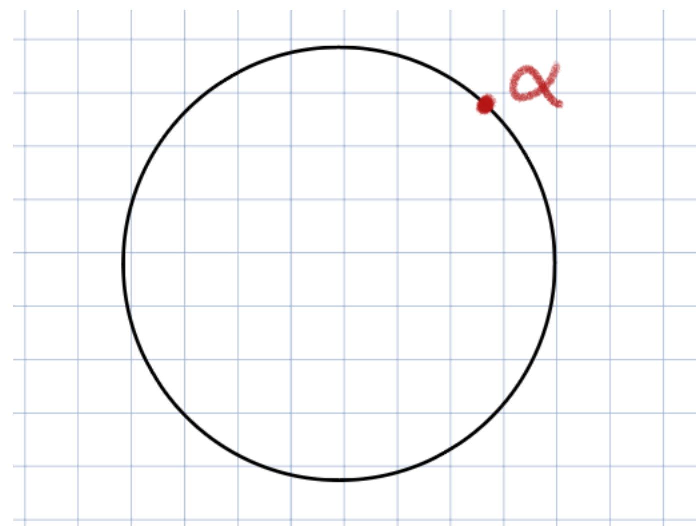
It should generalize the binary alphabet $\{+1, -1\}$

$X_{ij} \in ??$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

Unitary Matrices

- ± 1 are one-dimensional unitaries
- $X^*X = I$
- Eigenvalues of a unitary are $e^{i\theta}$



Eigenvalues are on the unit circle in the complex plane

Unitary Matrices

- ± 1 are one-dimensional unitaries
- $X^*X = I$
- Eigenvalues of a unitary are $e^{i\theta}$
- How about the set of unitaries with ± 1 eigenvalues as our alphabet?
    - Algebraically $X^*X = X^2 = I$
    - The only complex numbers satisfying these are ± 1
    - Terminology: Observables

Alphabet of the noncommutative CSP

Observables

$$X_{ij}^* X_{ij} = I$$

$$X_{ij}^2 = I$$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

Deterministic and Probabilistic Assignments

$x_{ij} \in \{+1, -1\}$

or

x_{ij} are binary-outcome random variables

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

Observables are operator generalizations of binary random variables (simplified)

$$X^*X = I \qquad X^2 = I$$

- ± 1-eigenspaces
- Probability of observing 1 is the normalized-dimension of $+1$-eigenspace
- $X = \Pi^+ - \Pi^-$
- Probability of observing 1 is $tr(\Pi^+)$
- tr is the dimension normalized trace

Difference: Observables and Random Variables (simplified)

- If x and y are independent binary r.v.'s then

$$Pr(x = 1, y = -1) = Pr(x = 1)Pr(Y = -1)$$

- If X and Y are commuting observables
  - Then probability of observing +1 and -1 when measuring X and Y simultaneously is

$$tr(\frac{I + X}{2}\frac{I - Y}{2})$$

- $\frac{I + X}{2}$ is the projection onto +1-eigenspace of X
- $\frac{I - Y}{2}$ is the projection onto -1-eigenspace of Y

Perfect Solution to the Noncommutative MagicSquare?

$X_{ij}^{*}X_{ij} = I$

$X_{ij}^{2} = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

Perfect Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Point2:

The algebraic nature of the alphabet in noncommutative CSPs cannot be ignored!

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Point2:

The algebraic nature of the alphabet in noncommutative CSPs cannot be ignored!

$X_{ij}^* X_{ij} = I$

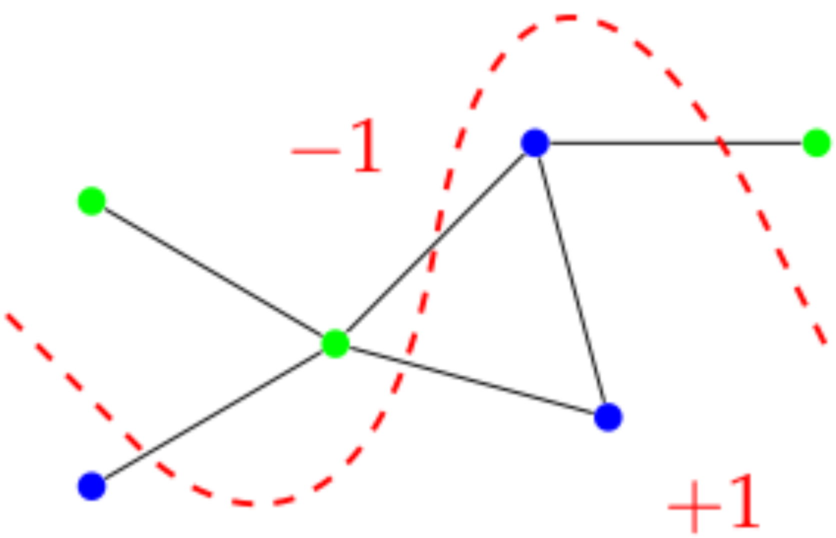
$X_{ij}^2 = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

$$X_{11}X_{12} = X_{12}X_{11}, \quad X_{12}X_{21} = -X_{21}X_{12}, \quad \cdots$$

Computational Aspects of Noncommutative CSPs

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

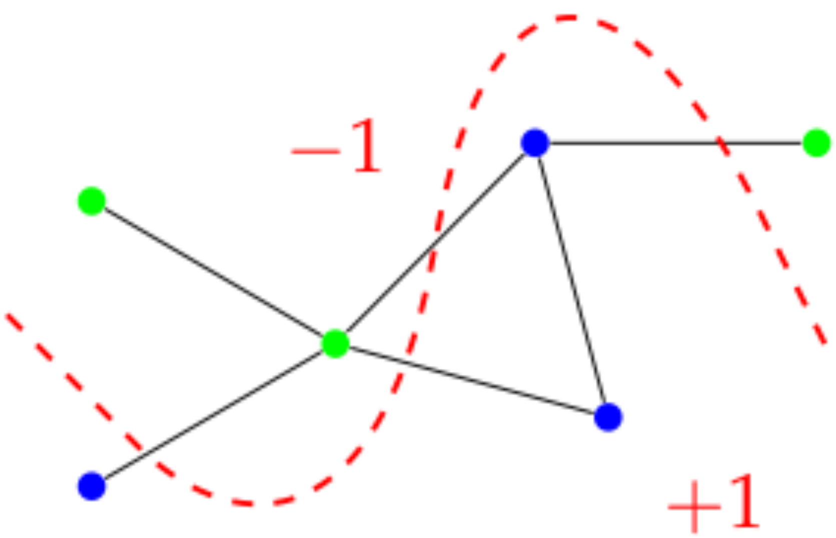
$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

Noncommutative Max-Cut

$$\max \quad \sum \frac{1 - X_i X_j}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

Max-Cut



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Noncommutative Max-Cut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

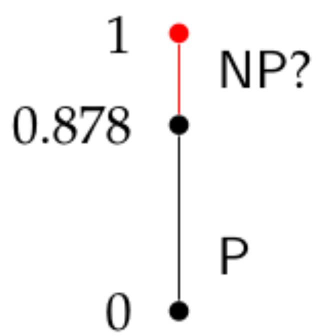
s.t. X_i is unitary with ± 1 eigenvalues

Hardness of generic NC-CSPs

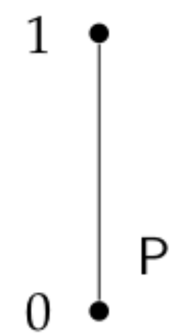
- Slofstra 2016: The exact value of NC-Label-Cover is uncomputable
- Ji, Natarajan, Vidick, Wright, Yuen 2020: Approximating it is also beyond reach
- Noncommutative analogue of the PCP theorem (Arora, Safra, Lund, Motwani, Sudan, Szegedy, Raz, Håstad)
  - PCP theorem: Approximating Label-Cover is NP-hard
  - NC-PCP theorem (MIP*=RE): Approximating NC-Label-Cover is RE-hard
- The day after PCP: approximability of other interesting CSPs
- Culf, M., Spirig: Approximation algorithms for noncommutative CSPs

Hardness of MaxCut

- Tsirelson 1980: NC-MaxCut is in P
- Karp 1972: Classical MaxCut is NP-hard



(a) Max-Cut



(b) Noncommutative Max-Cut

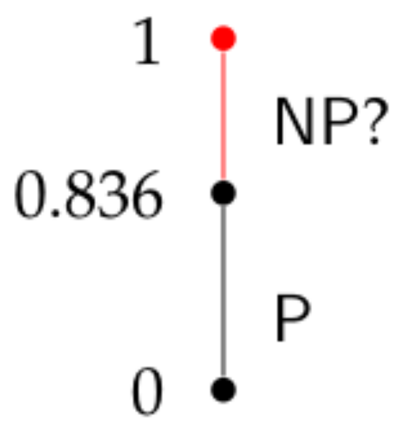
Best algorithm: Goemans and Williamson

Hardness: Khot, Kindler, Mossel, O'Donnell

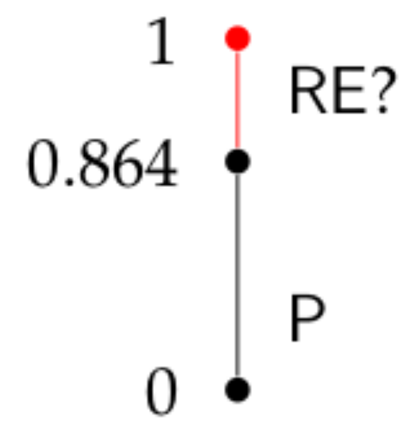
Algorithm: Tsirelson

Clifford algebra and SDPs

Hardness of Max-3-Cut



(a) Max-3-Cut



(b) Noncommutative Max-3-Cut

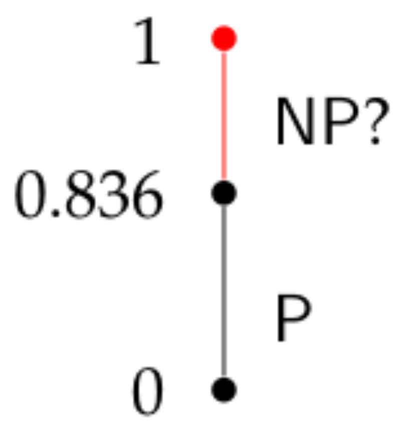
Algorithm: Frieze and Jerrum
Goemans and Williamson
de Klerk, Pasechnik, and Warners

Hardness: Khot, Kindler, Mossel, O'Donnell

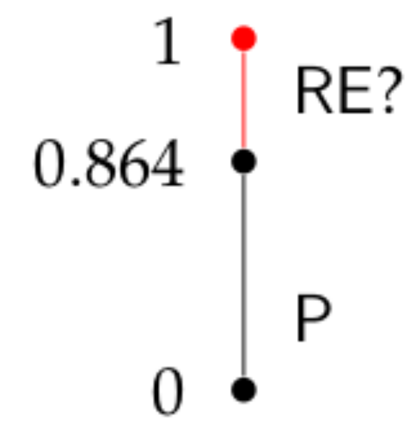
Algorithm: Culf, M., Spirig 2023

Hardness: Ji 2014

Hardness of Max-3-Cut



(a) Max-3-Cut



(b) Noncommutative Max-3-Cut

- Unique Games Conjecture (Khot) => Noncommutative Unique Games Conjecture (M., Spirig)
- Plurality Is Stablest Conjecture => ?

Recap of NC-CSPs

Recap of NC-CSPs

- One type of CSPs in quantum
- Very algebraic
  - Product of observables
  - Algebra generated by observables
  - Algebra of the optimal solution
- Physics: Quantum probability, quantum correlations
- Computer Science: PCP and UGC can be extended
  - Because alphabet retains its algebraic structure

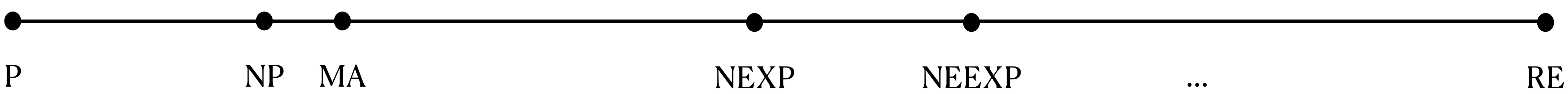
Recap of NC-CSPs: Capturing Computation

NC-CSPs:

Recap of NC-CSPs: Capturing Computation

NC-CSPs:

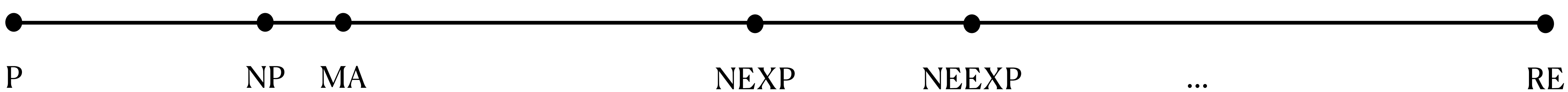
And much more:



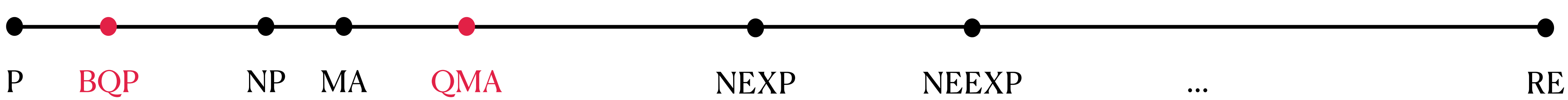
Recap of NC-CSPs: Capturing Computation

NC-CSPs:

And much more:



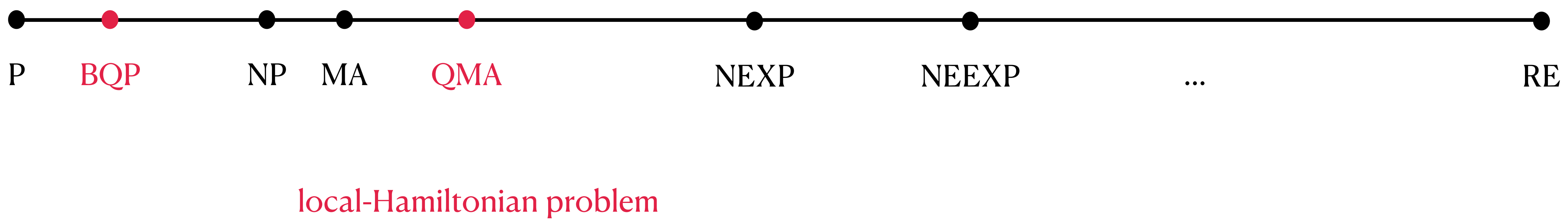
But it skips quantum computation!!



Quantum CSPs

a.k.a. local-Hamiltonians

Quantum-CSPs capture quantum computation



Assignments to Quantum CSPs: States

- Assignment to a CSP with n variables could be an element of $\mathbb{F}_2^n$
    - It is a vector space
    - It is an algebra
- Assignment to a quantum CSP with n qubits is a quantum state
- A state is a unit-norm vector in $\mathbb{C}^{2^n}$
    - Set of states is not an algebra
    - Not even a vector space
    - There is a binary operation: inner-product

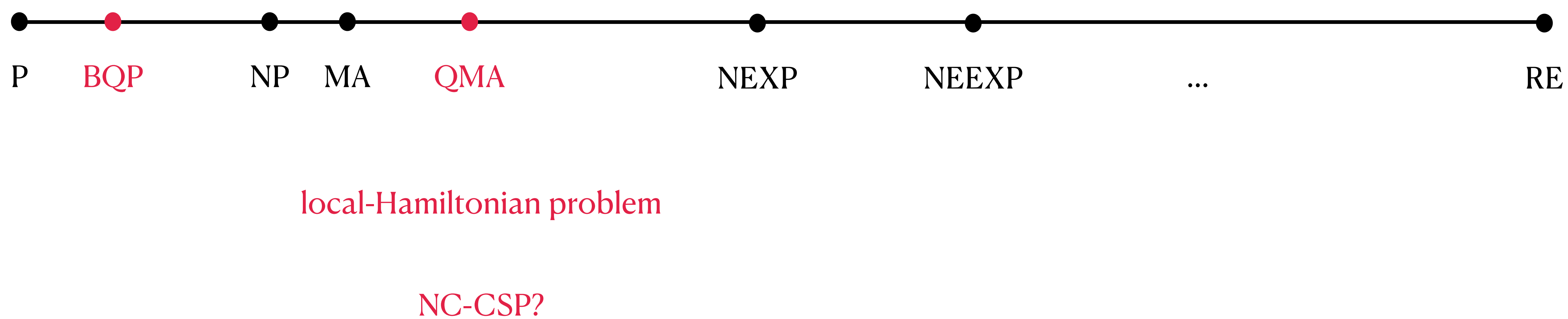
Assignments to Quantum CSPs: States

- Classical CSPs: $\mathbb{F}_2^n$
- Quantum CSPs: unit-norm vectors in $\mathbb{C}^{2^n}$ (states)
- $\mathbb{F}_2^n$ has a natural embedding into $\mathbb{C}^{2^n}$
  - $(0,0,\ldots,0,0)$, $(0,0,\ldots,0,1)$, ..., $(1,1,\ldots,1,1)$ in $\mathbb{F}_2^n$
  - $|0,0,\ldots,0,0>$, $|0,0,\ldots,0,1>$, ..., $|1,1,\ldots,1,1>$ in $\mathbb{C}^{2^n}$
- But any superposition of these basis vectors are also quantum states
  - $\alpha_1 |0,0,\ldots,0,0> + \alpha_2 |0,0,\ldots,0,1> + \cdots + \alpha_{2^n} |1,1,\ldots,1,1>$

    $|\alpha_1|^2 + |\alpha_2|^2 + \cdots + |\alpha_{2^n}|^2 = 1$

Open Problem

Is there a dual definition for BQP and QMA such that states are replaced by observables?



Quantum PCP Conjecture (the game version): For example see Natarajan and Nirkhe 2024

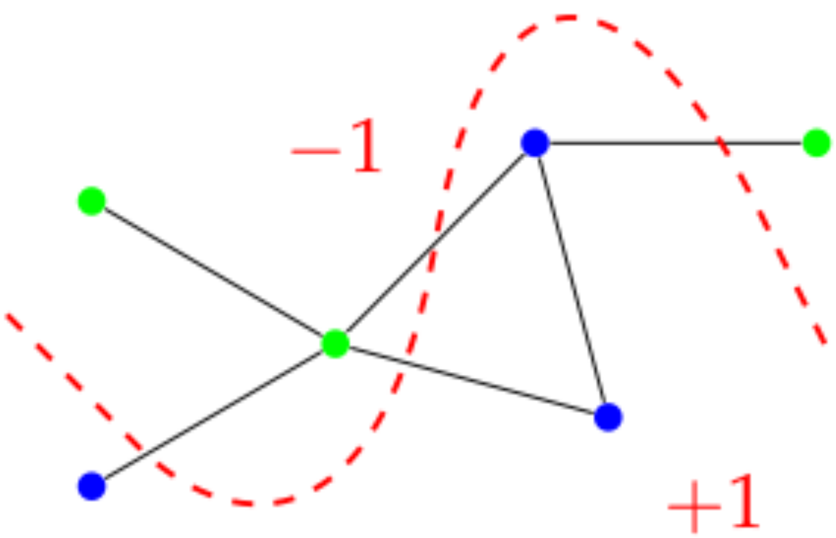
Argument against?

- $< u, v > = 0.5, \|u\|^2 = \|v\|^2 = 1$ are also algebraic relations
- But it only identifies the angle between the states
- But $XY = -YX, X^2 = Y^2 = 1$ are stronger:
  - Up to isomorphism identifies a group
  - The dihedral group of order 8
  - Any two unitaries of any dimension satisfying these relations must be isometrically equivalent (in some strong sense) to Pauli matrices

$$\sigma_x = \begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix} \text{ and } \sigma_y = \begin{bmatrix} 0 & -i \\ i & 0 \end{bmatrix}$$

Bonus 1: Quantum Correlations

Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

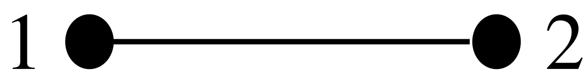
$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

Noncommutative Max-Cut

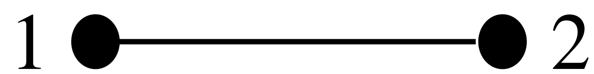
$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

Probabilistic Cut



Probabilistic Cut



$Pr_1(+1) = 0.8$ $\qquad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\qquad$ $Pr_2(-1) = 0.5$

Probabilistic Cut

$$Pr_1(+1) = 0.8 \qquad Pr_2(+1) = 0.5$$

$$Pr_1(-1) = 0.2 \qquad Pr_2(-1) = 0.5$$

Noncommutative Cut

Probabilistic Cut

$Pr_1(+1) = 0.8$ $\quad$ $Pr_2(+1) = 0.5$

$Pr_1(-1) = 0.2$ $\quad$ $Pr_2(-1) = 0.5$

Noncommutative Cut

$$Pr_{12}(+1, +1) = 0.1$$
$$Pr_{12}(+1, -1) = 0.2$$
$$Pr_{12}(-1, +1) = 0.3$$
$$Pr_{12}(-1, -1) = 0.4$$

Noncommutative Cut

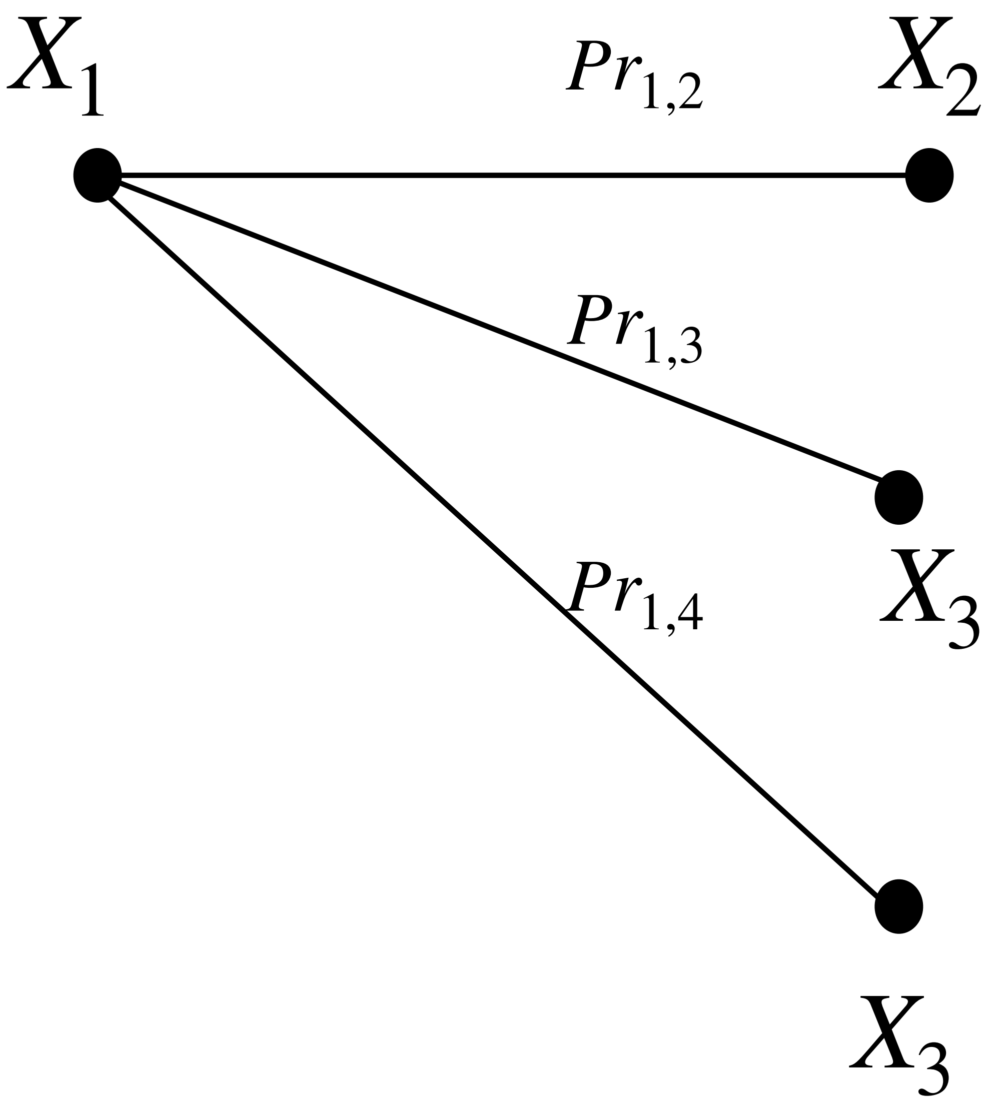
$$Pr_{12}(+1, +1) = tr\left(\frac{I + X_1}{2} \frac{I + X_2}{2}\right)$$

$$Pr_{12}(+1, -1) = tr\left(\frac{I + X_1}{2} \frac{I - X_2}{2}\right)$$

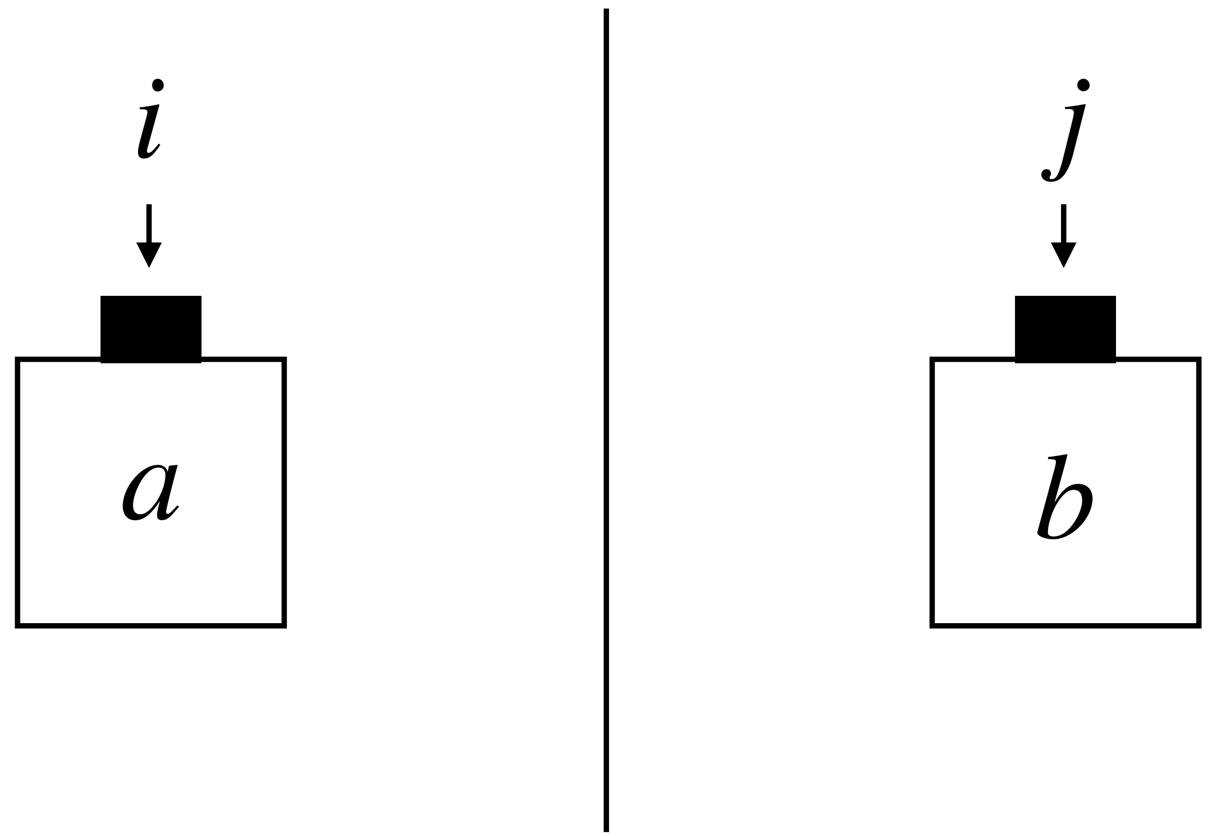
$$Pr_{12}(-1, +1) = tr\left(\frac{I - X_1}{2} \frac{I + X_2}{2}\right)$$

$$Pr_{12}(-1, -1) = tr\left(\frac{I - X_1}{2} \frac{I - X_2}{2}\right)$$

Inconsistencies of Edge Probabilities



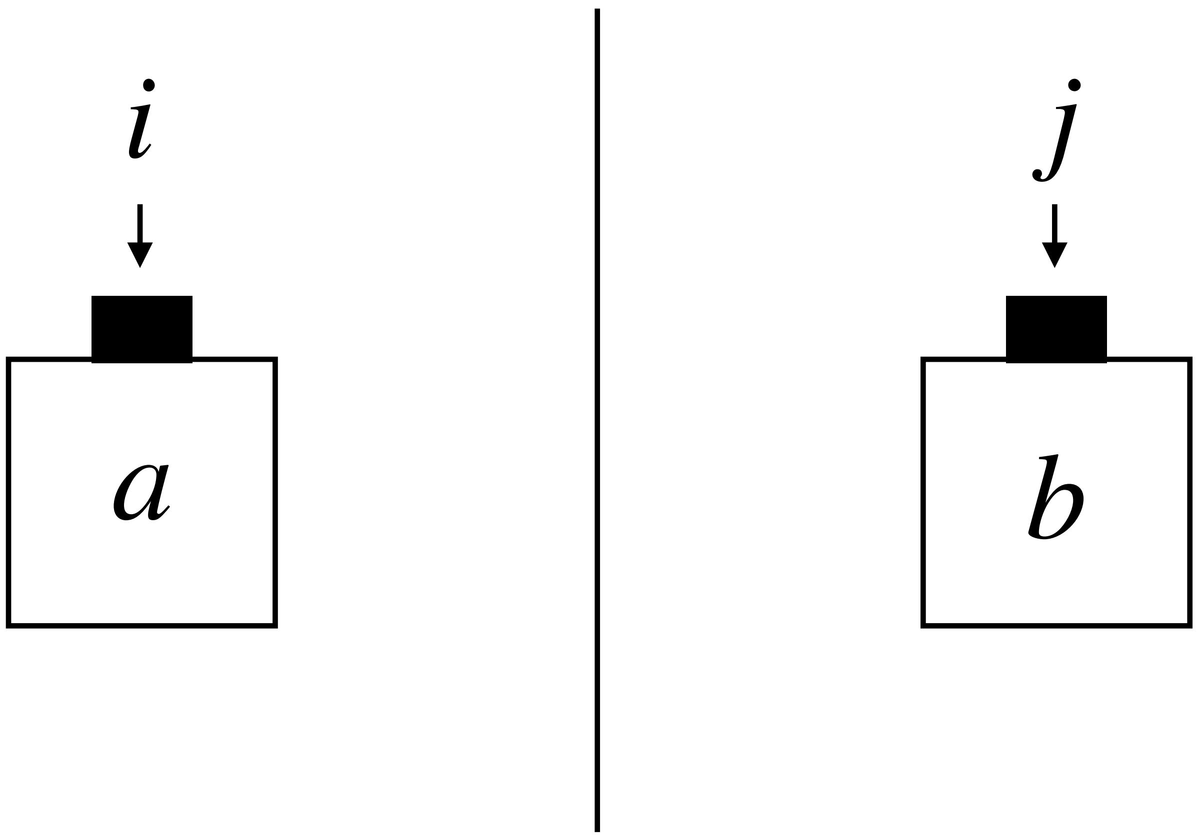
Operational Interpretation of Noncommutative Cuts



$i, j \in V,$

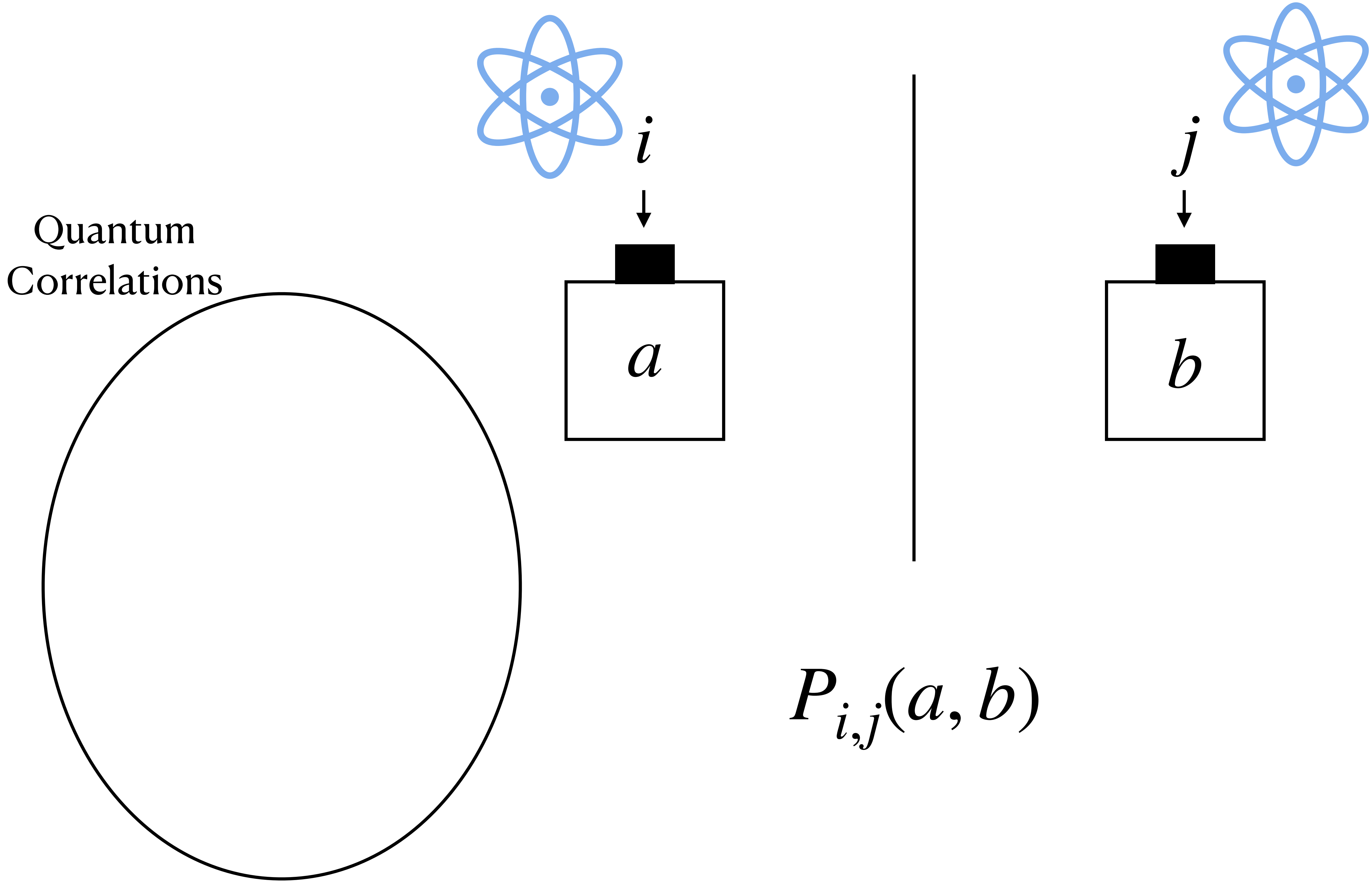
$a, b \in \{+1, -1\}$

Correlations



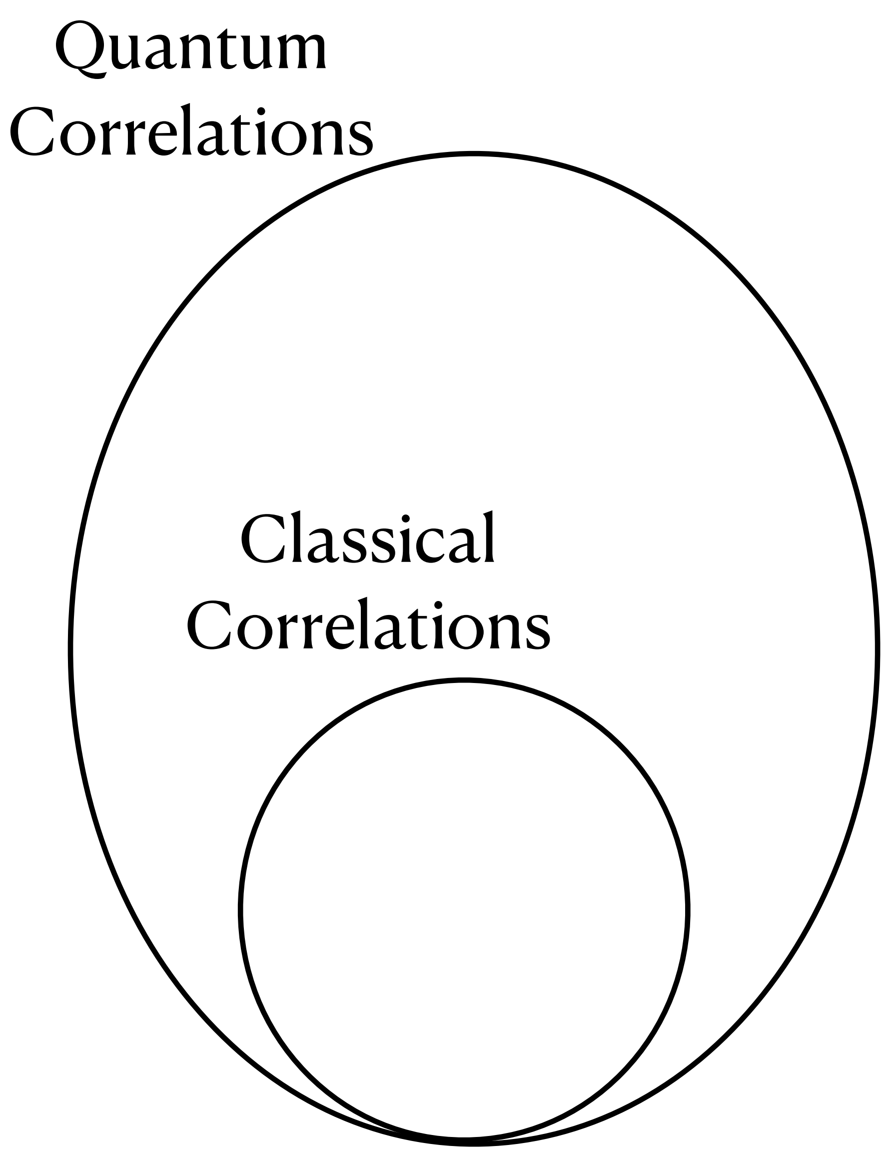
$$P_{i,j}(a, b)$$

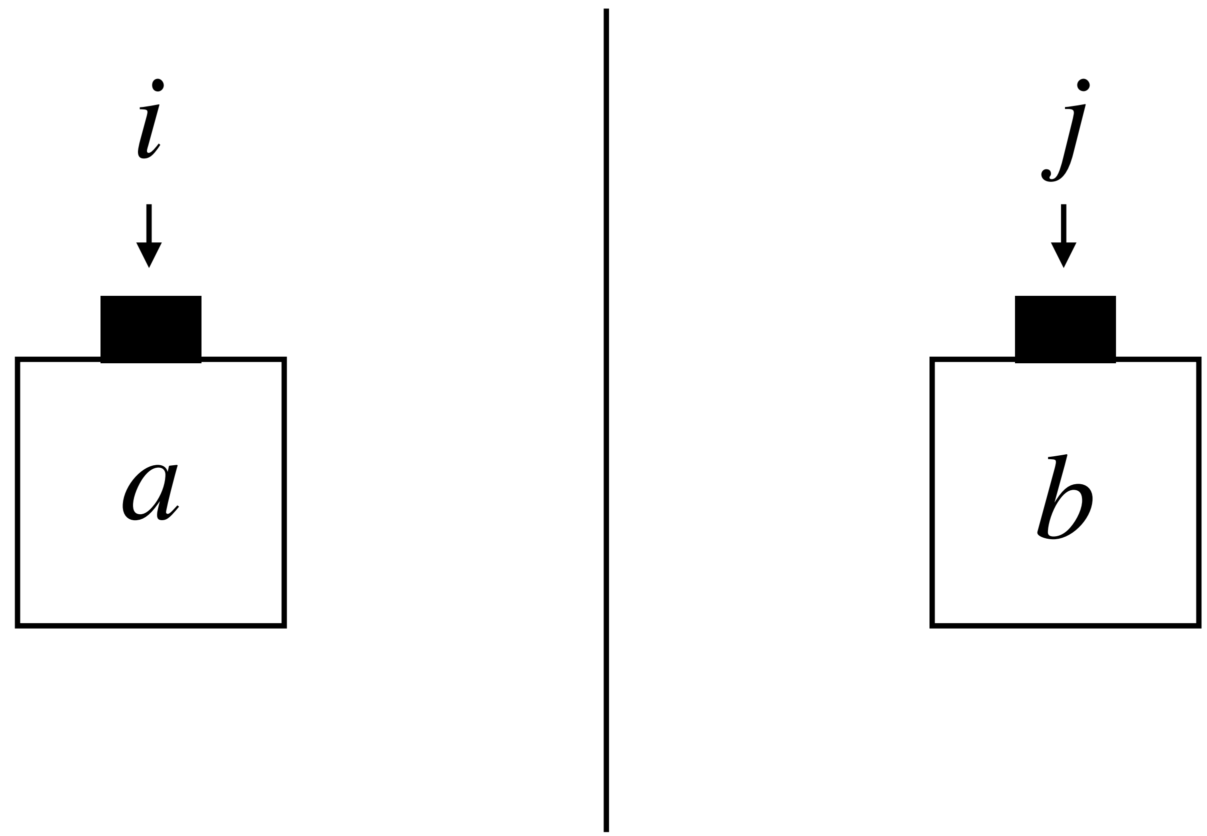
Quantum Correlations



$$P_{i,j}(a, b)$$

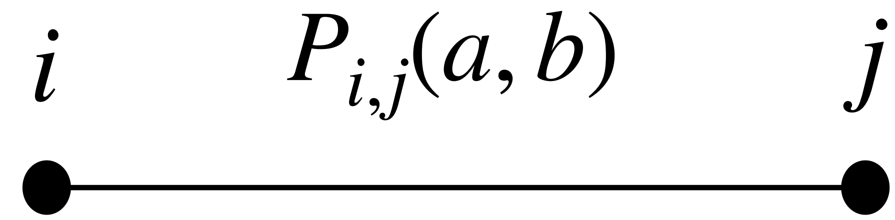
Classical Correlations





$$P_{i,j}(a, b)$$

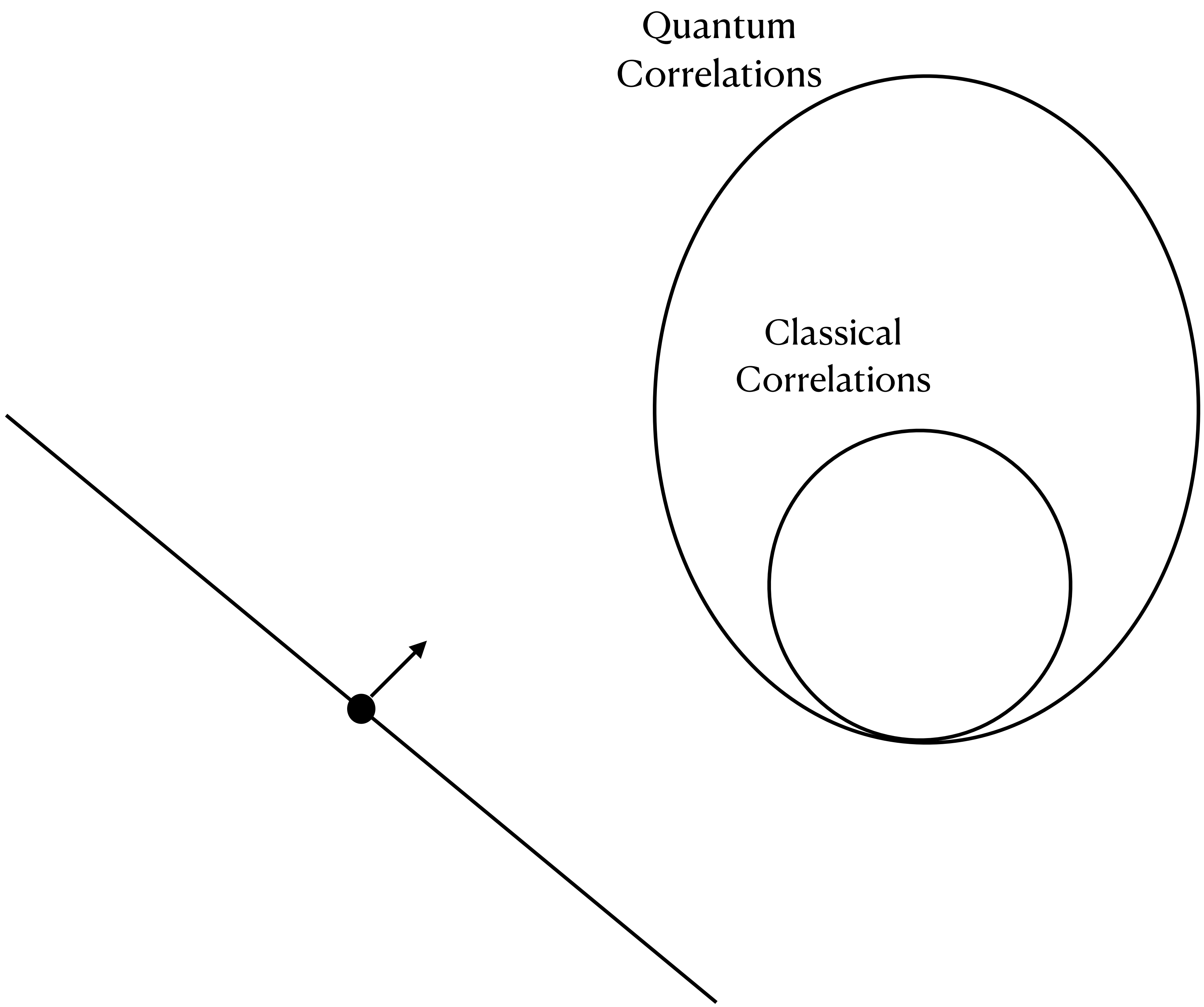
Edge Probabilities



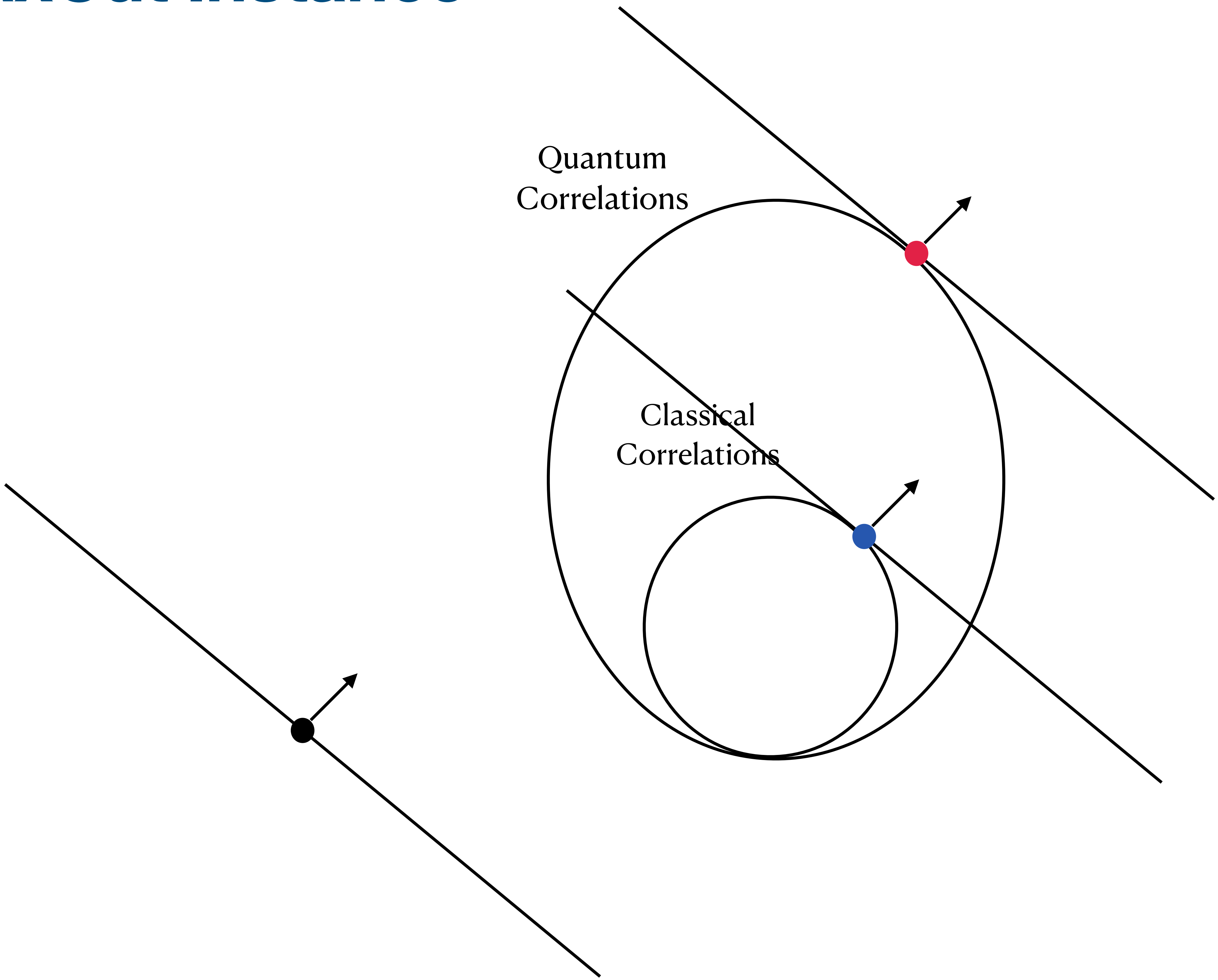
Quantum Correlations = Noncommutative Cuts

Classical Correlations = Probabilistic Cuts

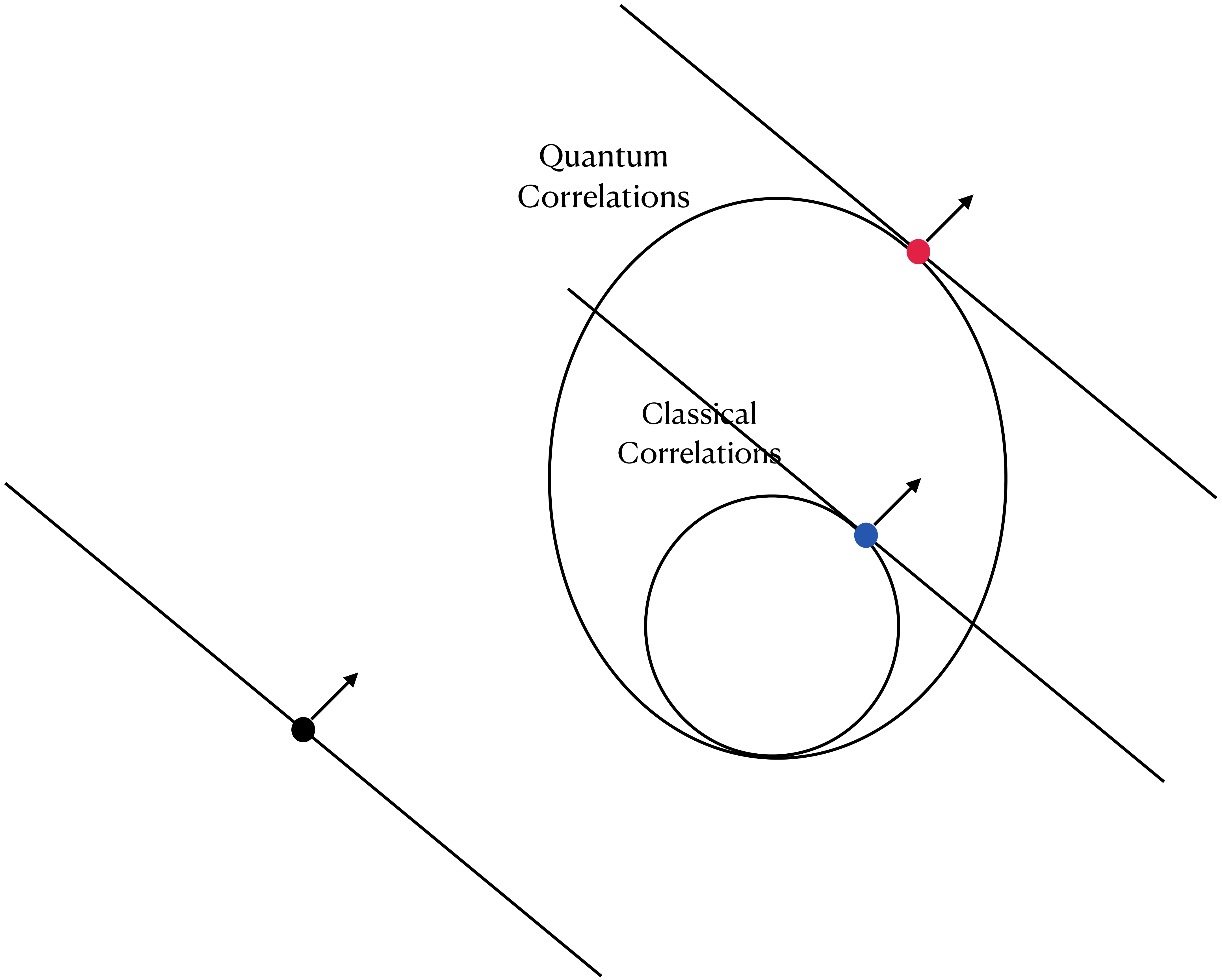
MaxCut Instance



MaxCut Instance



The 2022 Nobel Prize in Physics awarded to Alain Aspect, John F. Clauser, and Anton Zeilinger



Bonus Slides 2

Concepts: Anticommuting Algebras and Relative Distributions

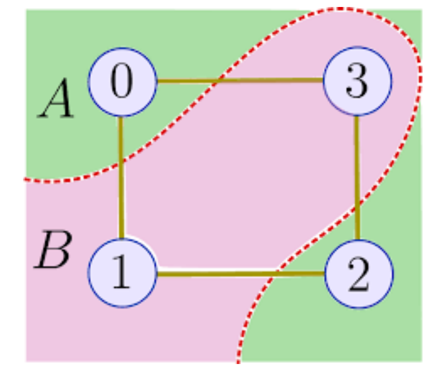
- Hyperplane rounding of Goemans-Williamson $\vec{r} = (r_1, \ldots, r_n)$
- A random operator $R = r_1\sigma_1 + \cdots + r_n\sigma_n$
- σ_i's generate generalized Weyl-Brauer algebra

Concepts: Anticommuting Algebras and Relative Distributions

- Given a λ, sample unitaries U, V uniformly such that $< U, V > = \lambda$
- Sample eigenvalues α, β from U, V
- What is the angle between α, β?
- It is the well-known Cauchy distribution

Proof that NC-MaxCut is easy

Goemans-Williamson



Max-Cut

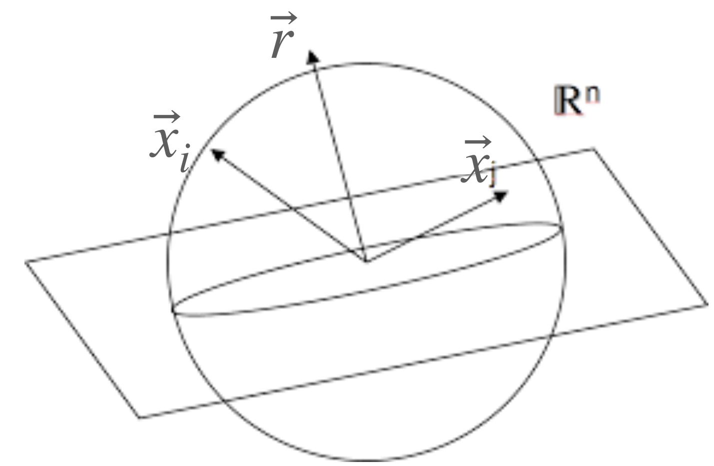
$$\max \sum \frac{w_{ij}}{2}(1 - x_i x_j)$$

s.t. $x_i^2 = 1$

Max-Cut-SDP

$$\max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

s.t. $\langle \vec{x}_i, \vec{x}_i \rangle = 1$



Goemans-Williamson Theorem

$0.878 \times$ **Max-Cut-SDP** $\leq$ **Max-Cut** $\leq$ **Max-Cut-SDP**

Hyperplane rounding scheme

Sample vector $\vec{r}$ from the unit sphere

Let x_i be the sign of $\langle \vec{r}, \vec{x}_i \rangle$

$$\mathbb{E}\left(\frac{1 - x_i x_j}{2}\right) \geq 0.878 \frac{1 - \langle \vec{x}_i, \vec{x}_j \rangle}{2}$$

Tsirelson's theorem

NC-Max-Cut

$$\max Tr \sum \frac{w_{ij}}{2}(1 - X_i X_j)$$

s.t. $X_i^2 = X_i^* X_i = 1$

Tsirelson's theorem

NC-Max-Cut = Max-Cut-SDP

Max-Cut-SDP

$$\max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

s.t. $\langle \vec{x}_i, \vec{x}_i \rangle = 1$

Max-Cut: Proof relies on anticommutation

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s.t. $X_i^2 = X_i^* X_i = 1$

Tsirelson's theorem

NC-Max-Cut = Max-Cut-SDP

- We need a bunch of operators $\sigma_1, \ldots, \sigma_n$ such that for every vector $\vec{x} = (x_1, \ldots, x_n)$ such that $||\vec{x}|| = 1$
  - Operator $X = x_1\sigma_1 + \cdots + x_n\sigma_n$ is unitary $X^*X = 1$ and $X^2 = 1$
  - And if $Y = y_1\sigma_1 + \cdots + y_n\sigma_n$ is another operator, it holds that $\langle X, Y \rangle = \langle \vec{x}, \vec{y} \rangle$

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- For these to hold it is necessary and sufficient that σ_i are unitary and order-2 and they are pairwise anticommuting $\sigma_i\sigma_j = -\sigma_j\sigma_i$

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- This was the relation of optimal operators in CHSH

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- Similarly, in the optimal solution of NC-Max-Cut we have $X_iX_j + X_jX_i = \lambda_{ij}I$