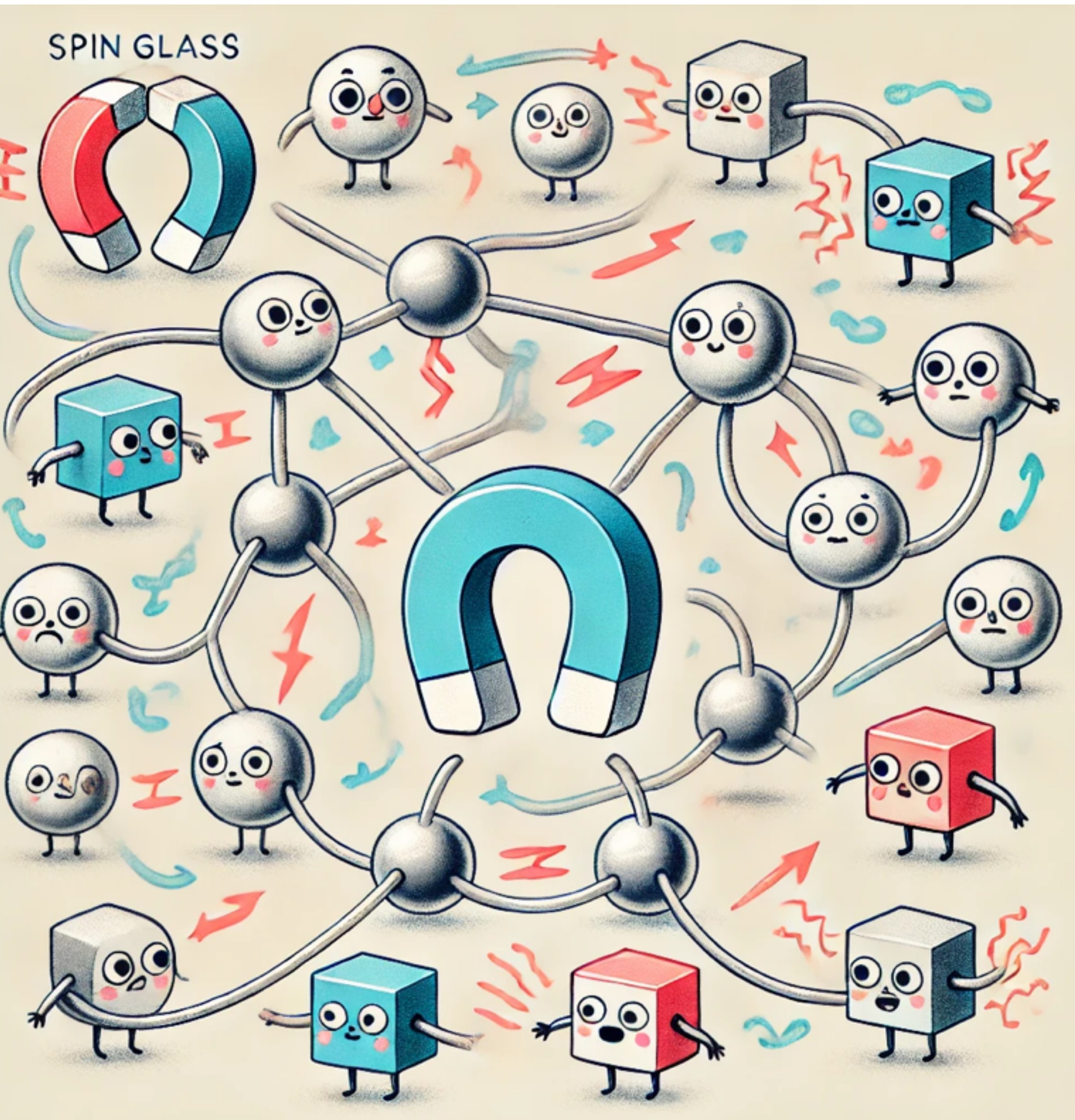
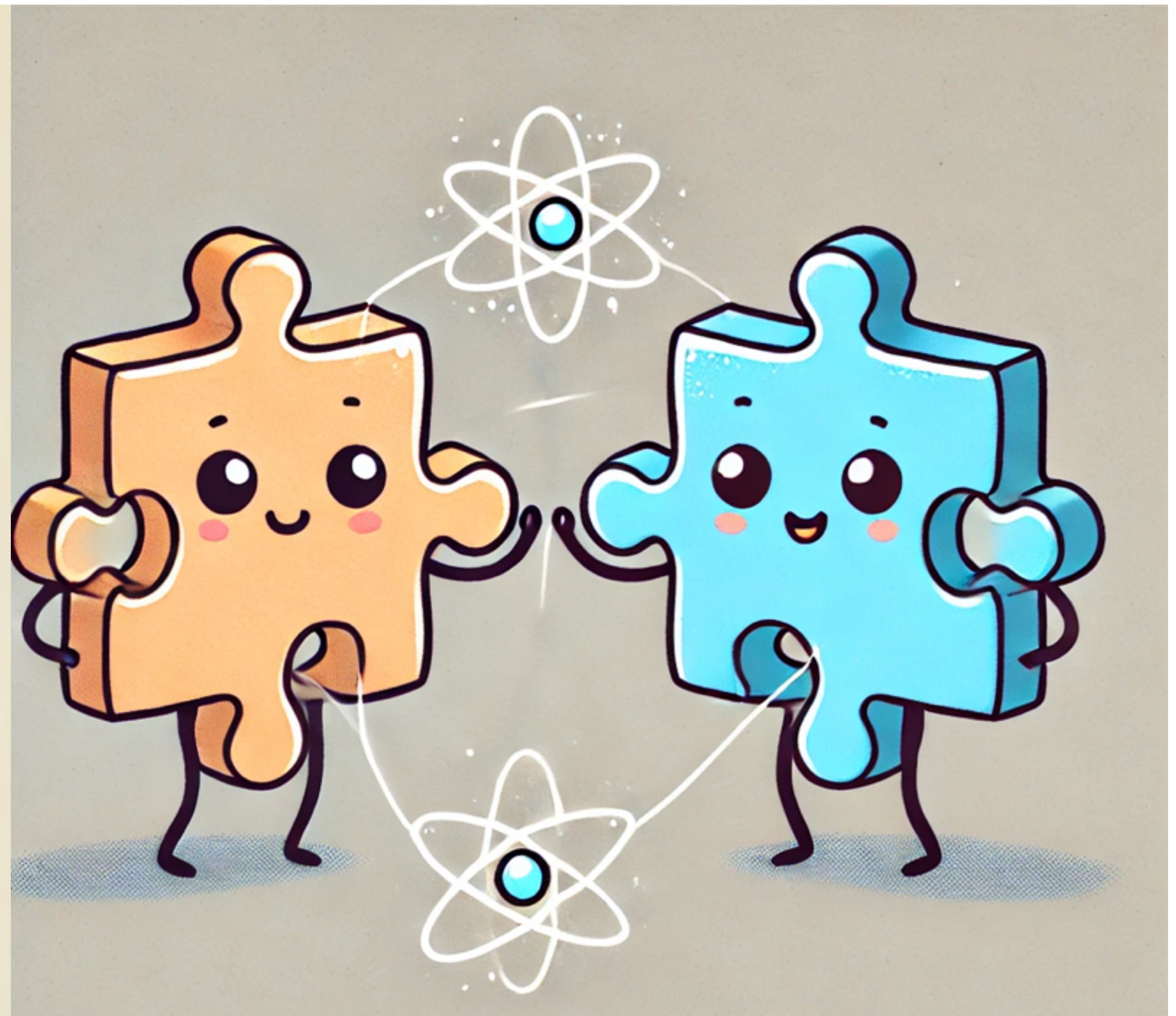
Constraint Satisfaction in the Quantum World

Algebras, CSPs, and Quantum Computing





Hamoon Mousavi (Simons Institute, Berkeley)

Constraint satisfaction problems

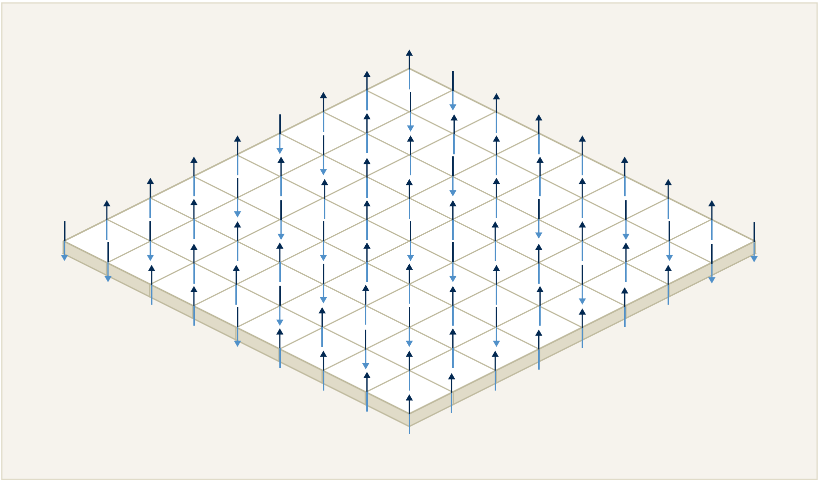
$$\begin{cases} a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\ a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\ \vdots \\ a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m, \end{cases}$$

System of equations

Constraint satisfaction problems

$$
\begin{cases}
a_{11}x_1 + a_{12}x_2 + \cdots + a_{1n}x_n = b_1 \\
a_{21}x_1 + a_{22}x_2 + \cdots + a_{2n}x_n = b_2 \\
\vdots \\
a_{m1}x_1 + a_{m2}x_2 + \cdots + a_{mn}x_n = b_m,
\end{cases}
$$

System of equations



Ising model

Constraint satisfaction problems

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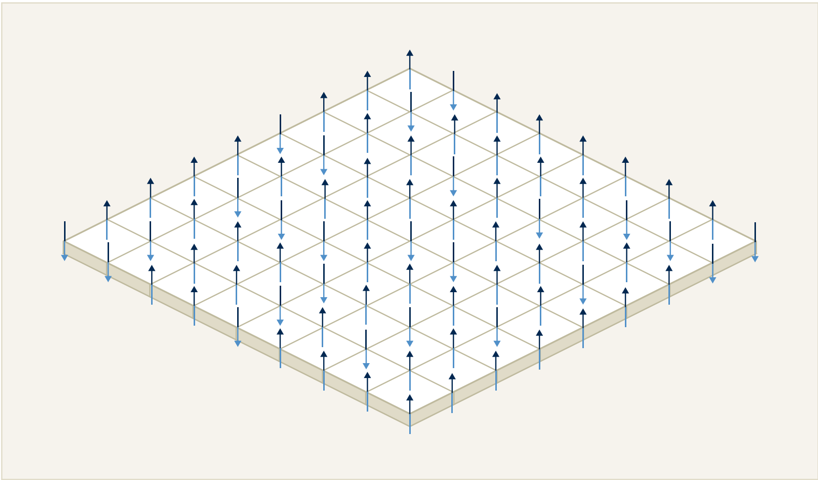
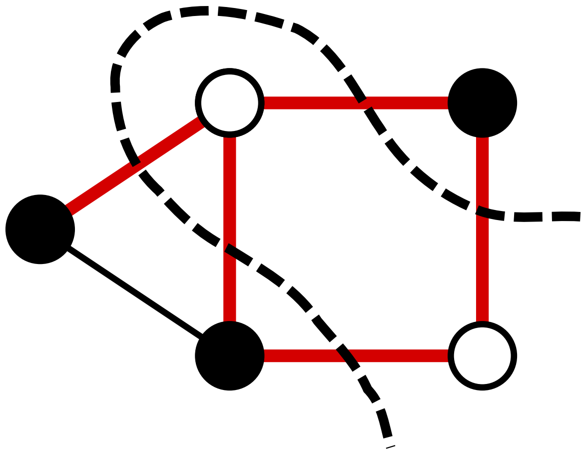
System of equations

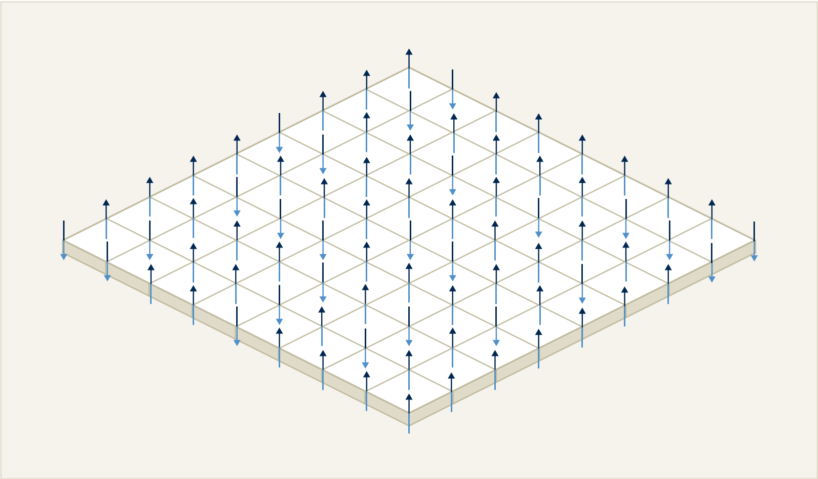


Ising model



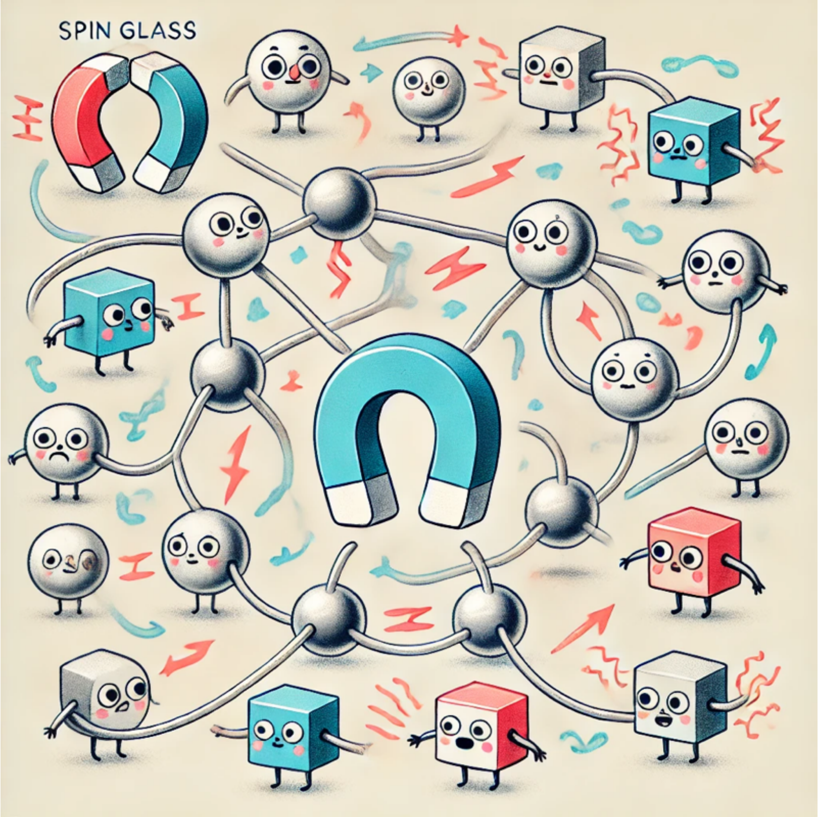
MaxCut

Constraint satisfaction problems

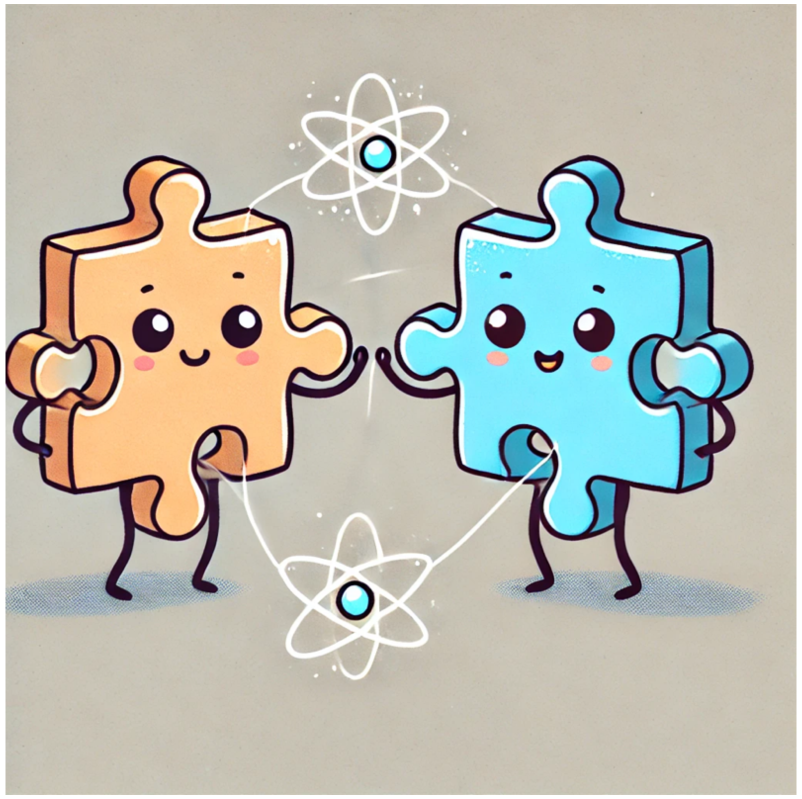


Ising model

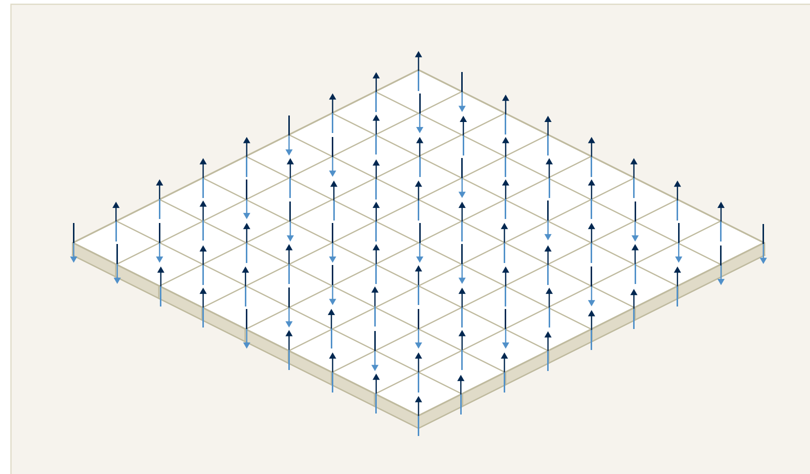
Quantum Ising model



XOR games

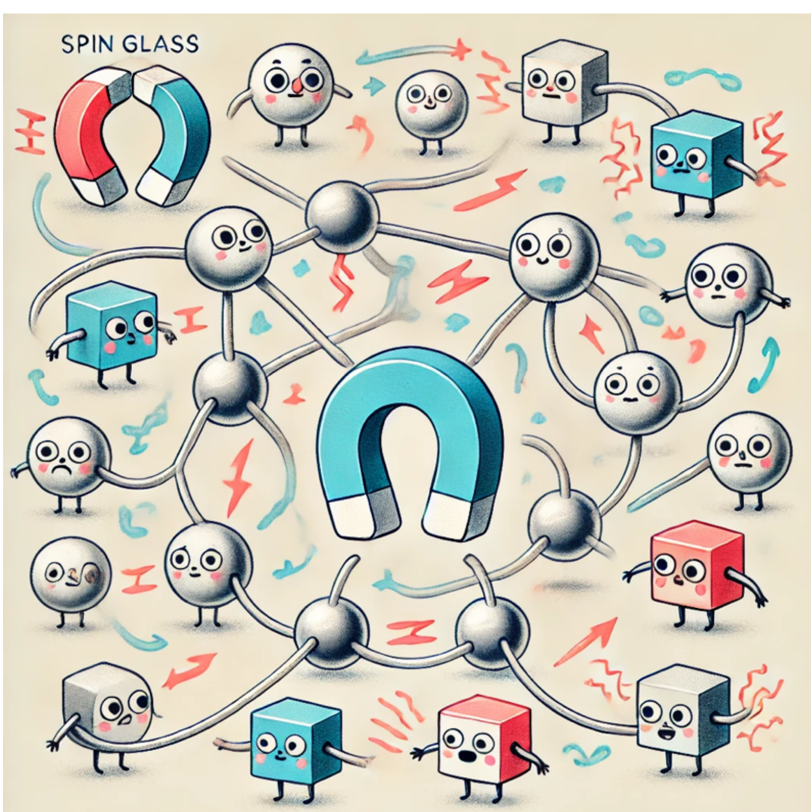


Constraint satisfaction problems

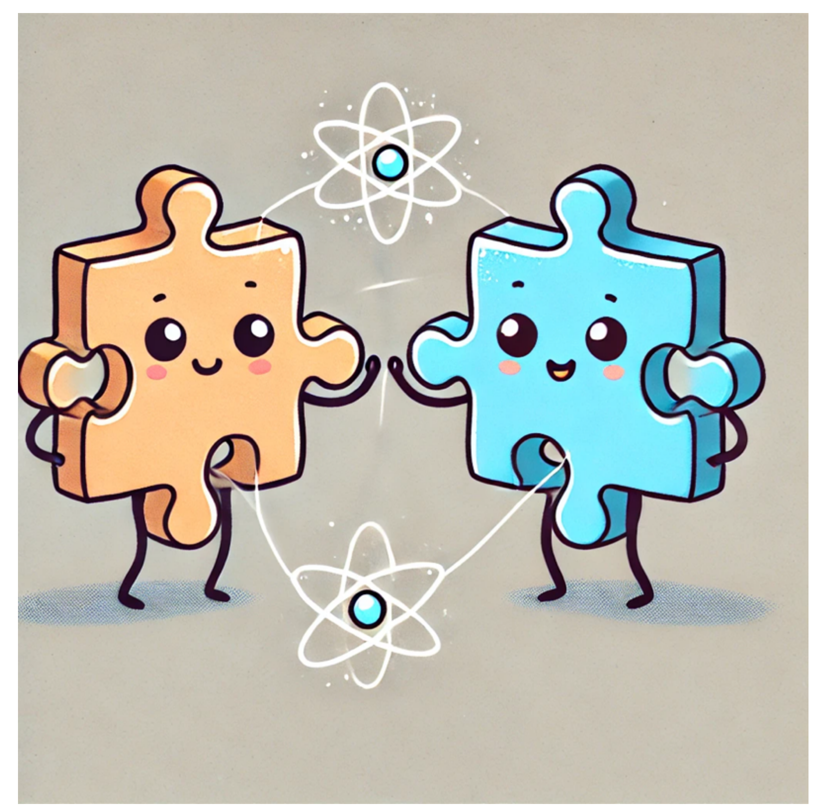


Ising model

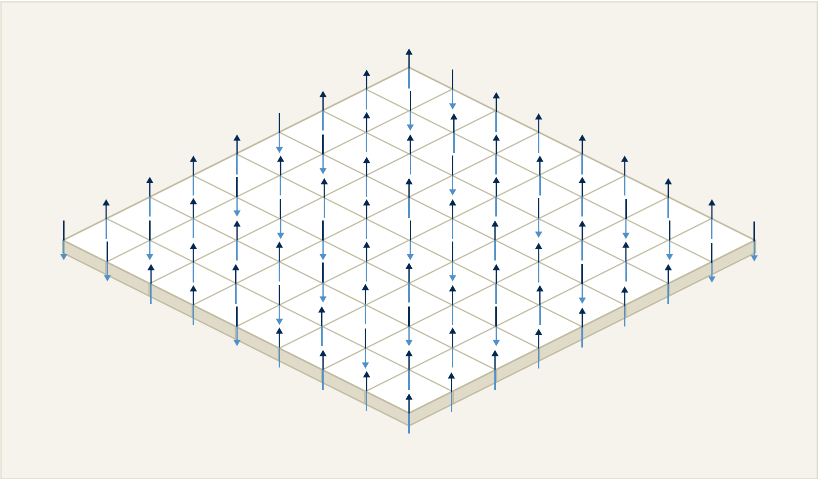
Quantum Ising model (local Hamiltonians)



XOR games (nonlocal games, MIP*,...)

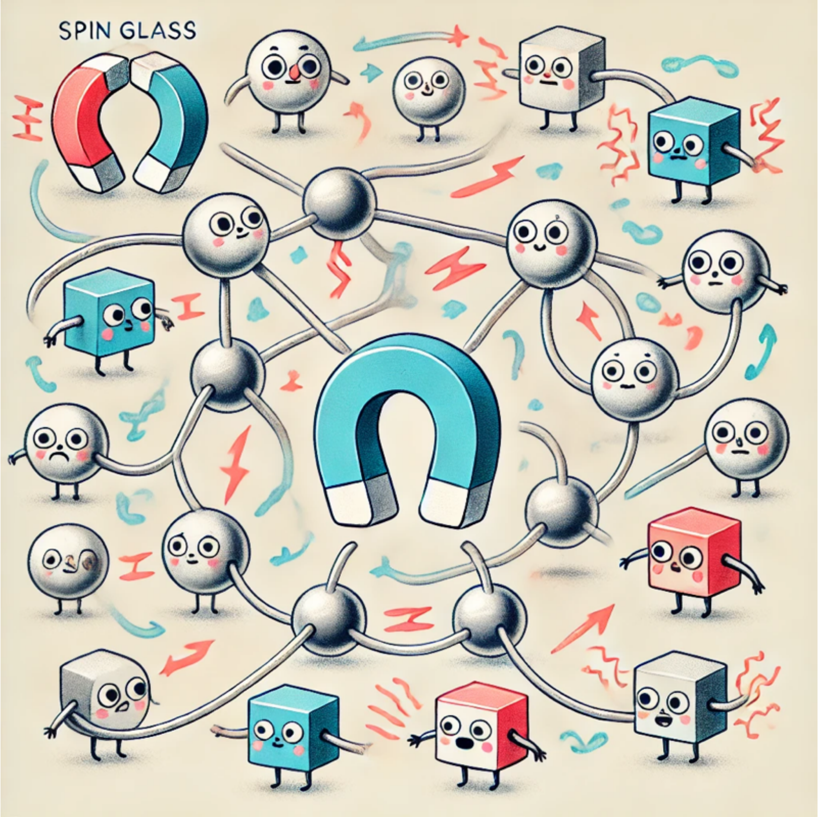


Approximation algorithms for constraint satisfaction problems

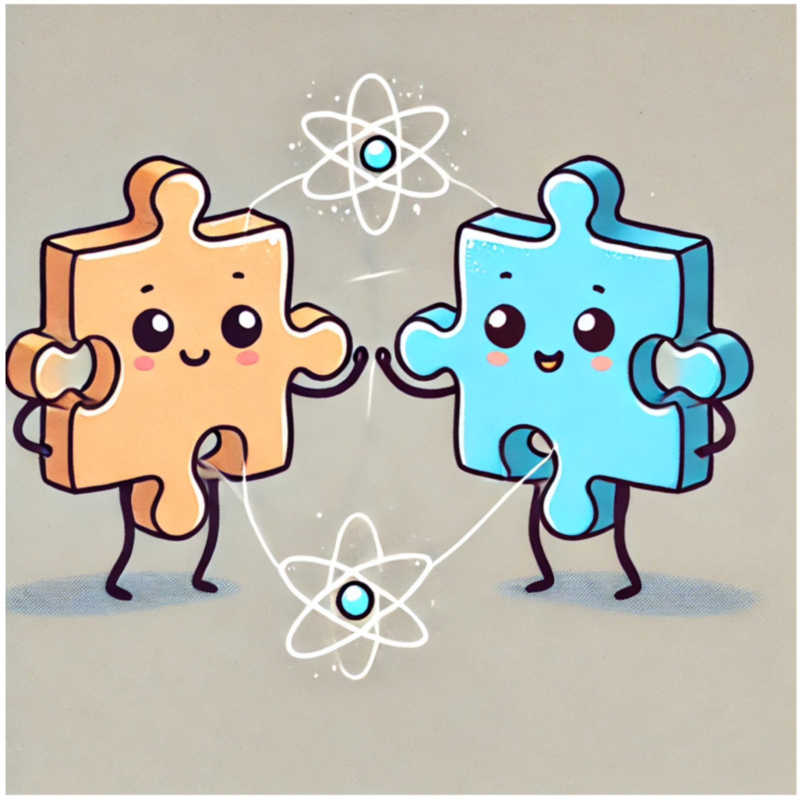


Classical CSPs (constraint satisfaction problems)

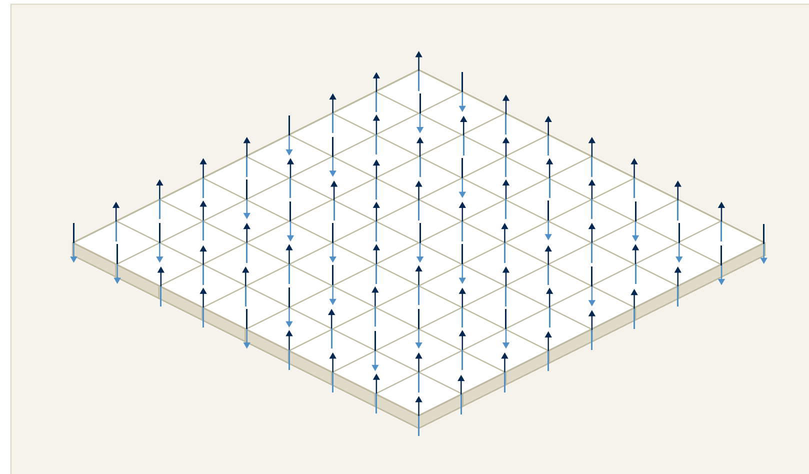
Local Hamiltonians



Nonlocal games



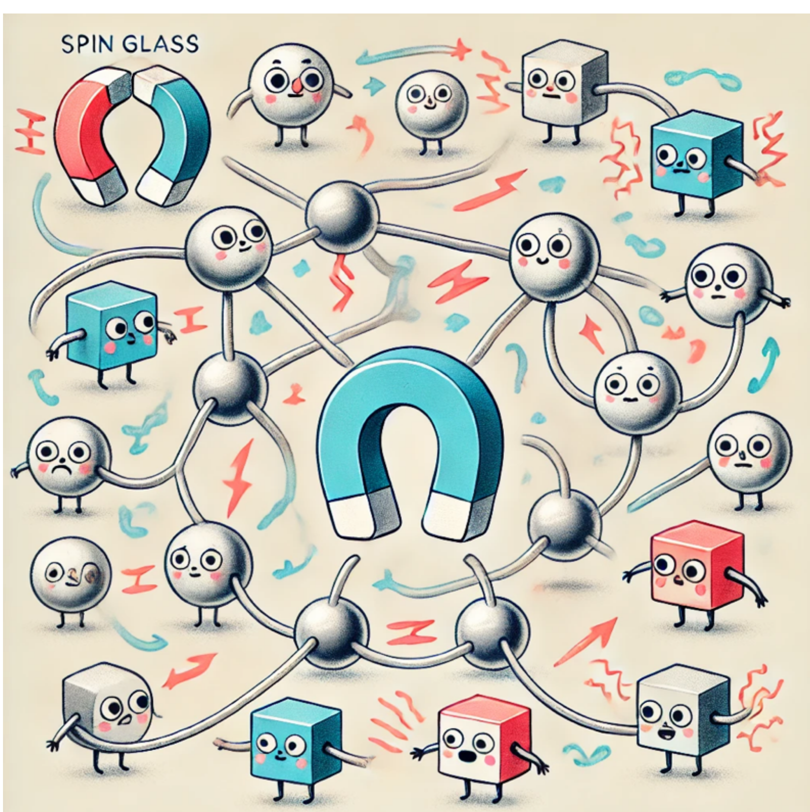
Approximation algorithms for constraint satisfaction problems



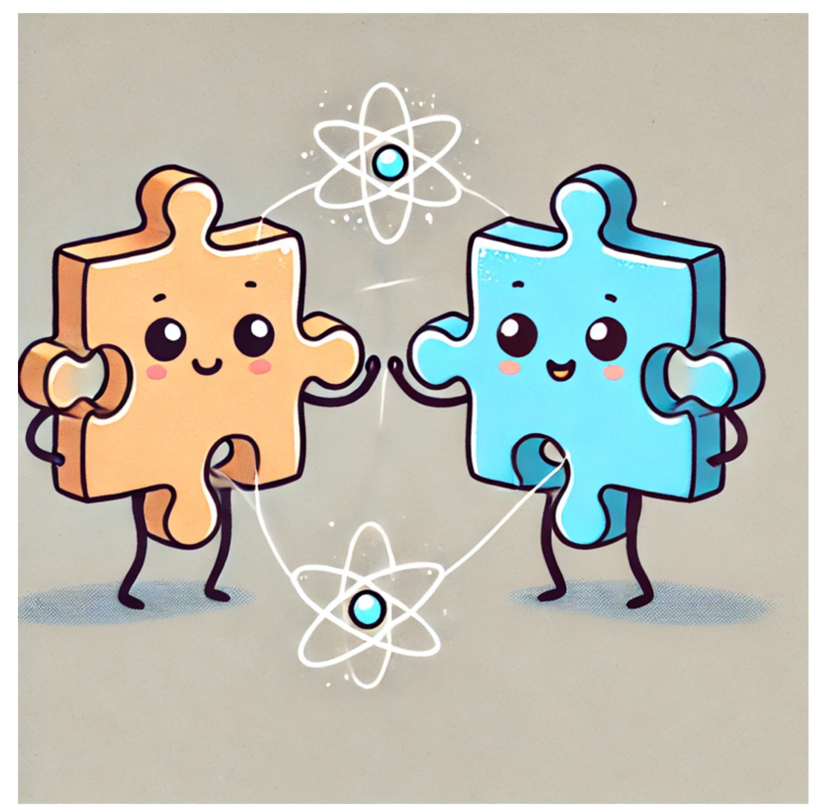
Classical CSPs (constraint satisfaction problems)

A rich extension

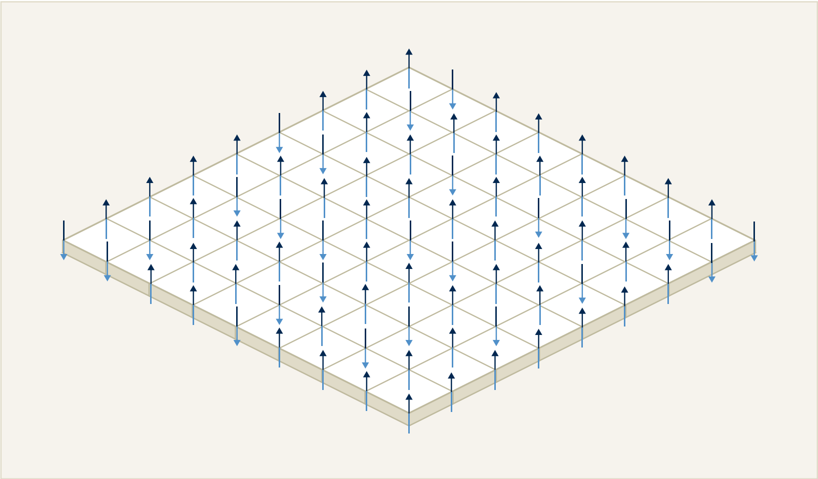
Local Hamiltonians



Nonlocal games



Approximation algorithms for constraint satisfaction problems

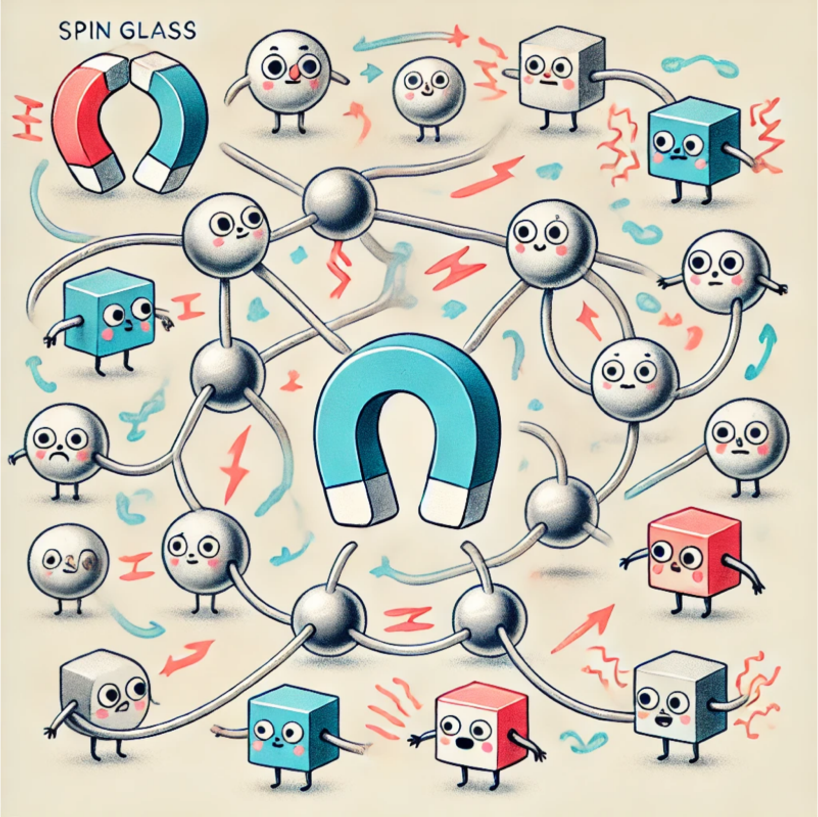


Classical CSPs (constraint satisfaction problems)

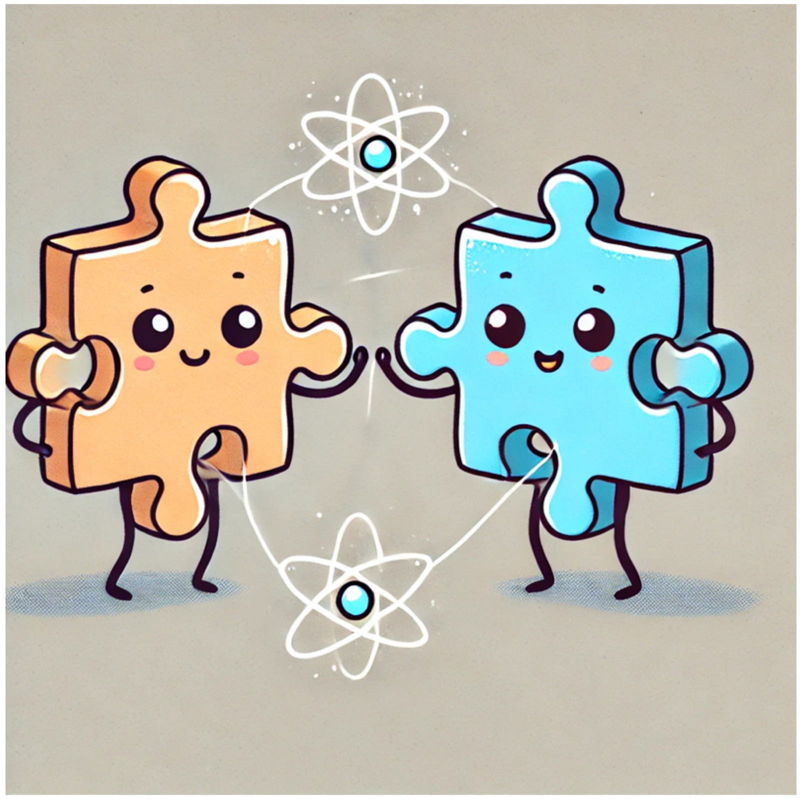
Does not extend!

A rich extension

Local Hamiltonians



Nonlocal games



Approximation algorithms for constraint satisfaction problems

Classical CSPs (constraint satisfaction problems)

Does not extend!

A rich extension

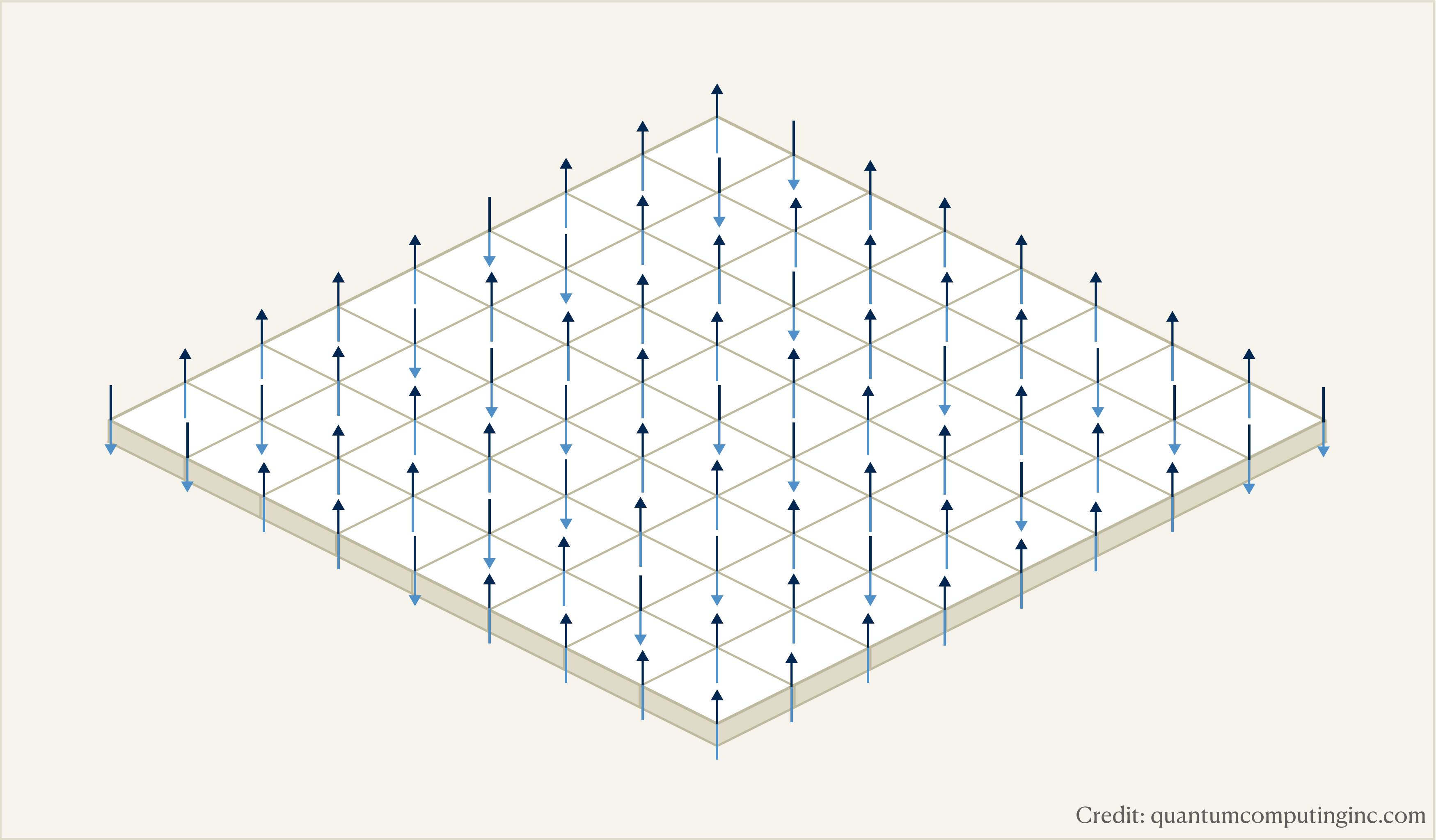
Local Hamiltonians

Nonlocal games

Classical CSPs

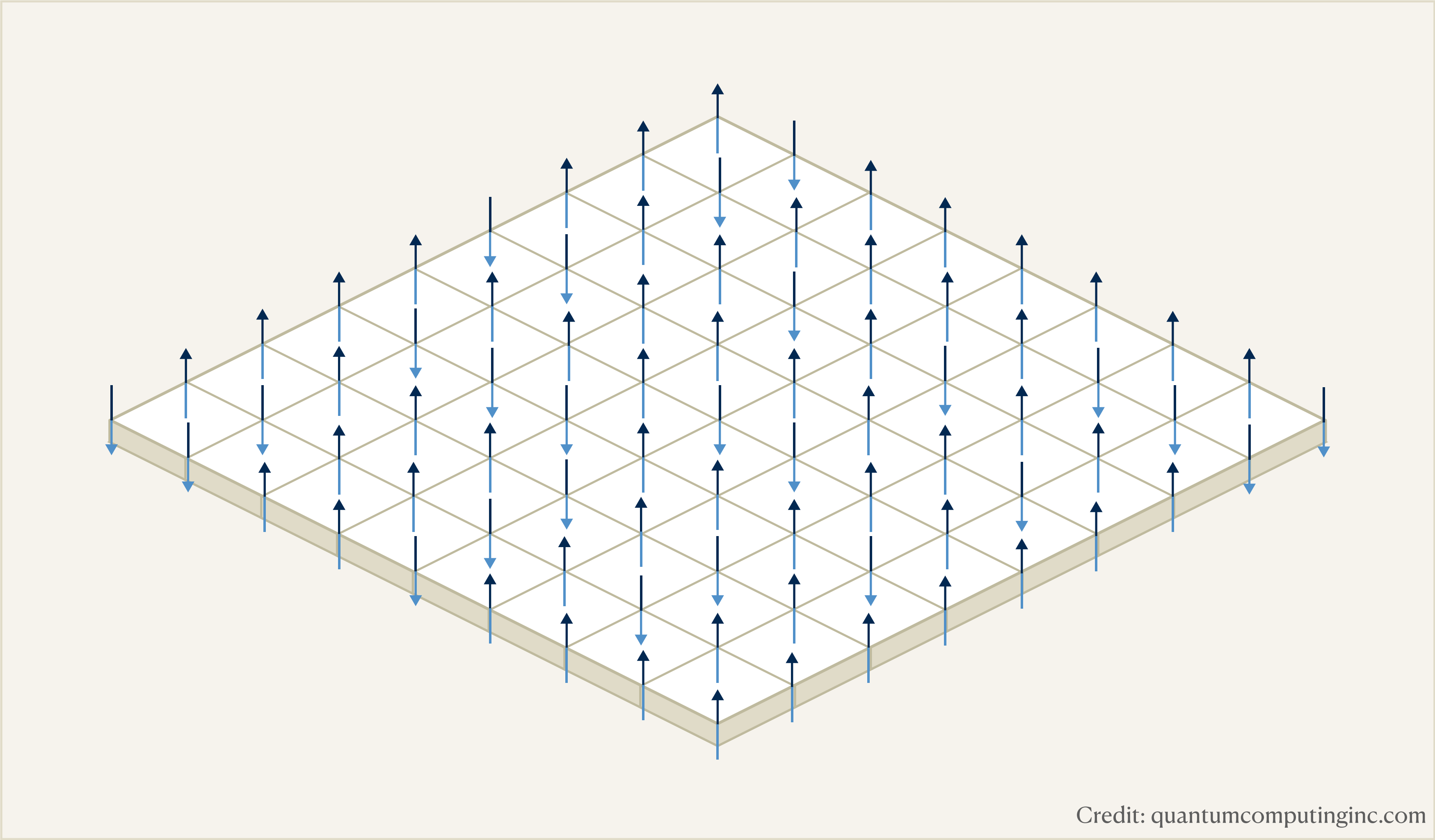
Ising model

Ising model



$$E = \sum_{i,j} J_{ij} x_i x_j \qquad x_i \in \{-1, +1\}$$

Ising model



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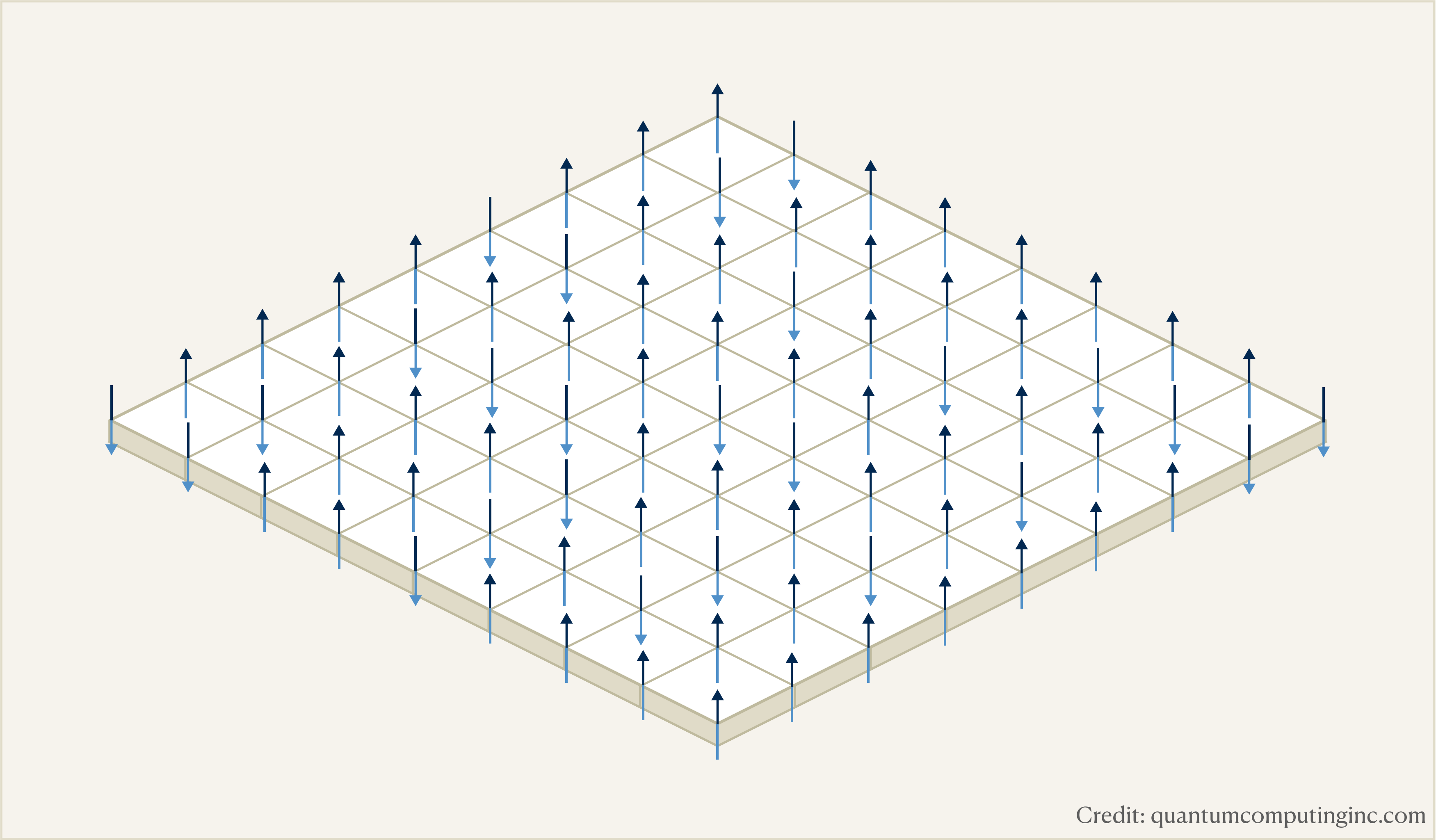
Ising model



Minimize $\sum_{i,j} x_i x_j$

$x_i \in \{-1, +1\}$

Ising model

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

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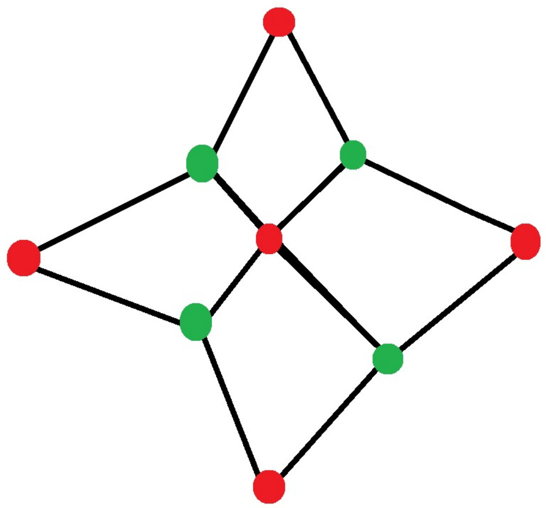

Ising model

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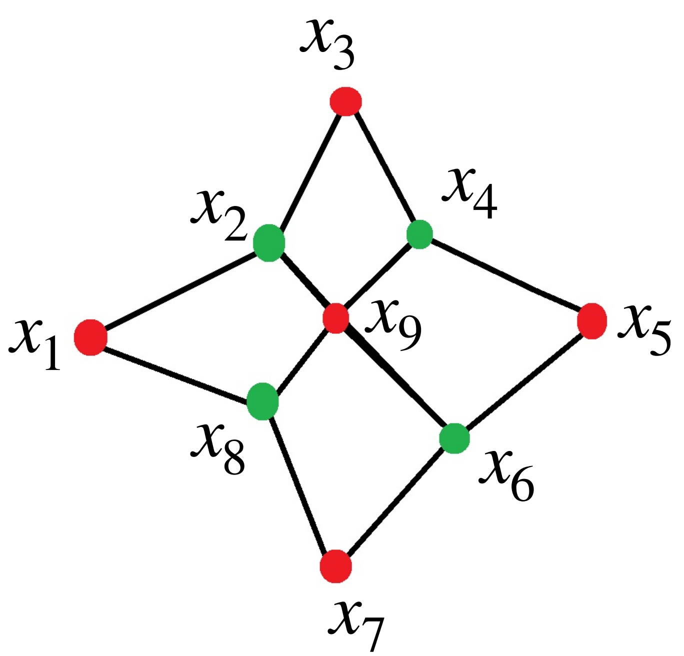
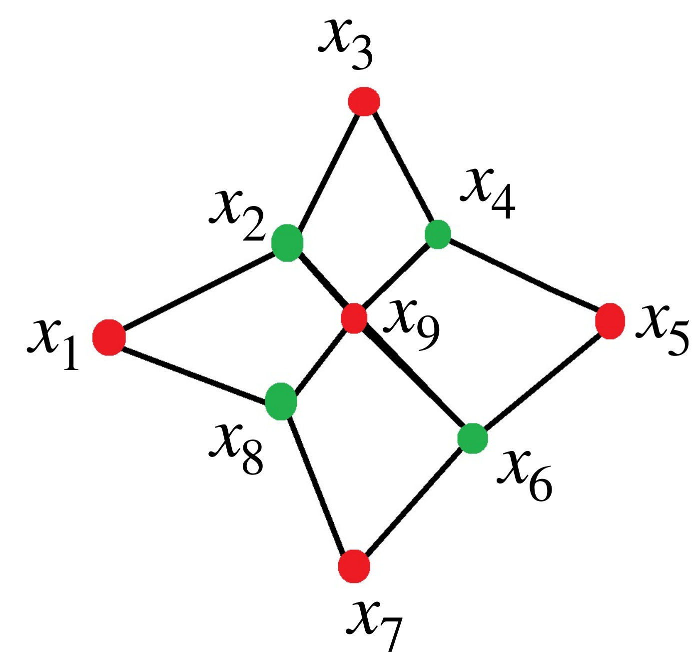


2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$



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Subject to: $x_i \in \{-1, +1\}$

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Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Higher-dimensional relaxation

Minimize $\sum_{i,j} \langle \vec{x}_i, \vec{x}_j \rangle$

Subject to: $d \in \mathbb{N}$

$\vec{x}_i \in \mathbb{R}^d$

$\|\vec{x}_i\| = 1$

2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Higher-dimensional relaxations

Minimize $\sum_{i,j} \langle \vec{x}_i, \vec{x}_j \rangle$

Subject to: $d \in \mathbb{N}$

$\vec{x}_i \in \mathbb{R}^d$

$\|\vec{x}_i\| = 1$

Minimize $\sum_{i,j} \langle X_i, X_j \rangle$

Subject to: $d \in \mathbb{N}$

$X_i \in \mathbb{C}^{d \times d}$

2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Higher-dimensional relaxations

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?

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Higher-dimensional relaxations

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Subject to: $d \in \mathbb{N}$

$\vec{x}_i \in \mathbb{R}^d$

$\|\vec{x}_i\| = 1$

Minimize $\sum_{i,j} \langle X_i, X_j \rangle$

Subject to: $d \in \mathbb{N}$

$X_i \in \mathbb{C}^{d \times d}$

X_i is a unitary operator with $\{-1, +1\}$ eigenvalues

2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Operator 2-coloring

Minimize $\sum_{i,j} \langle X_i, X_j \rangle$

Subject to: $d \in \mathbb{N}$

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X_i is a unitary operator with $\{-1, +1\}$ eigenvalues

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Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Operator 2-coloring

Minimize $\sum_{i,j} \langle X_i, X_j \rangle$ ← $\langle X_i, X_j \rangle = \frac{\mathrm{Tr}(X_i^\dagger X_j)}{d} = \mathrm{tr}(X_i^\dagger X_j)$

Subject to: $d \in \mathbb{N}$

$X_i \in \mathbb{C}^{d \times d}$

X_i is a unitary operator with $\{-1, +1\}$ eigenvalues

2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Operator 2-coloring

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: $d \in \mathbb{N}$

$X_i \in \mathbb{C}^{d \times d}$

X_i is a unitary operator with $\{-1, +1\}$ eigenvalues

2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Operator 2-coloring

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: $d \in \mathbb{N}$

$X_i \in \mathbb{C}^{d \times d}$

X_i is a unitary operator with $\{-1, +1\}$ eigenvalues ← Is called a (binary) observable in quantum info

2-coloring

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i \in \{-1, +1\}$

Operator 2-coloring

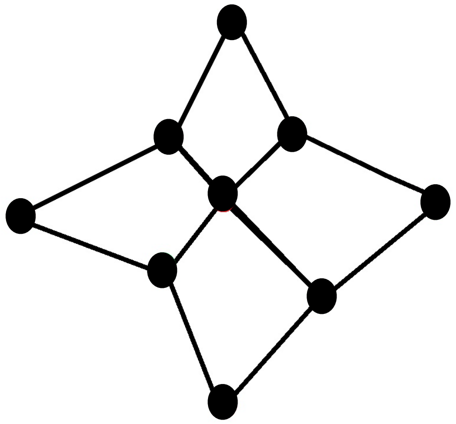
Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

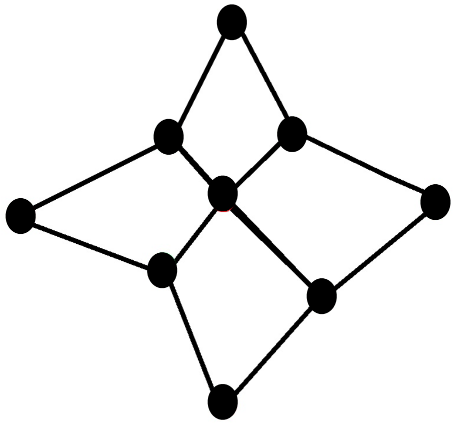
Subject to: X_i is an observable

Operator assignment

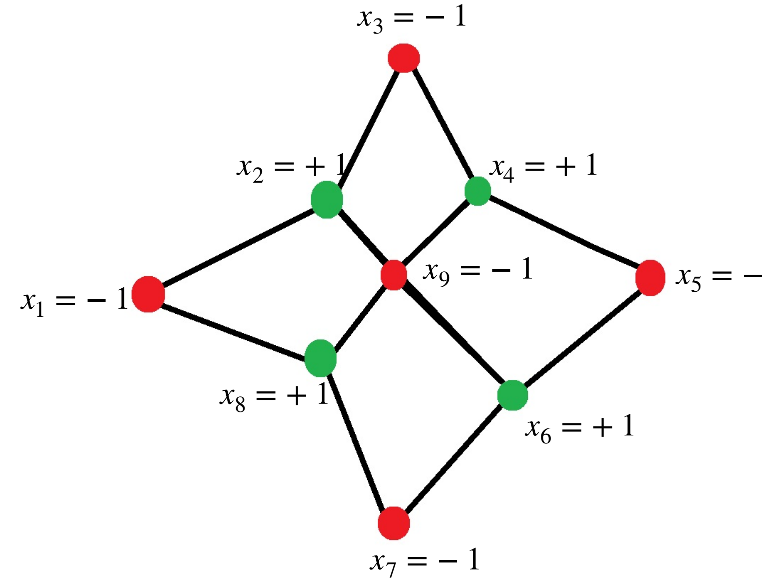
is a generalization of

random assignment

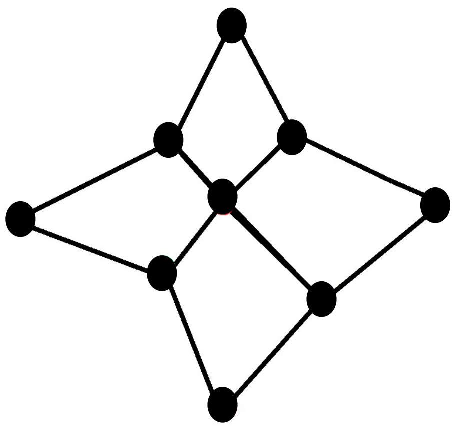




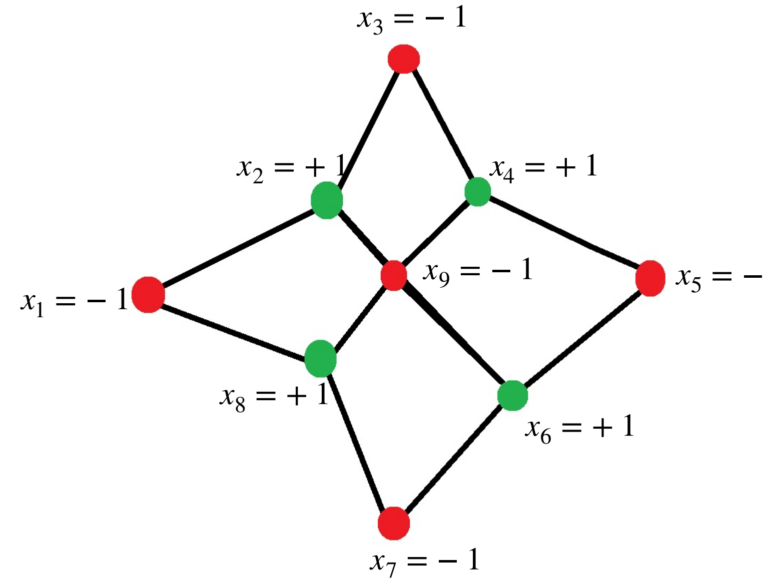
Deterministic:



$$\sum_{ij} x_i x_j$$



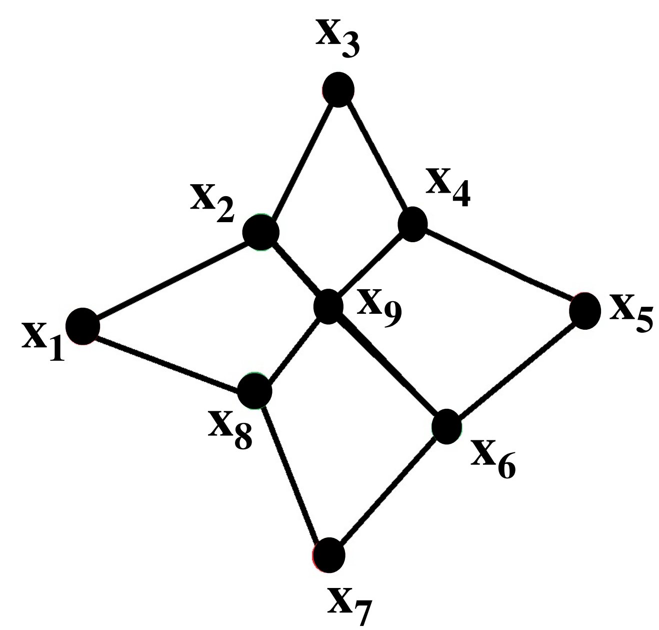
Deterministic:



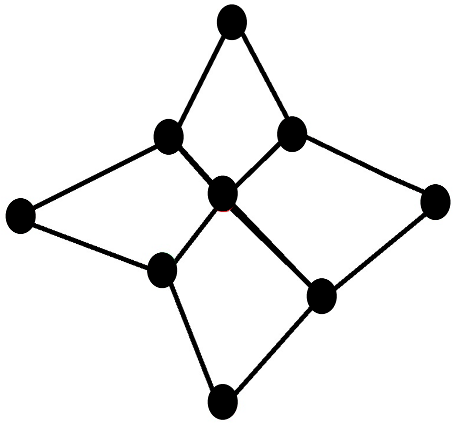
$$\sum_{ij} x_i x_j$$

Probabilistic:

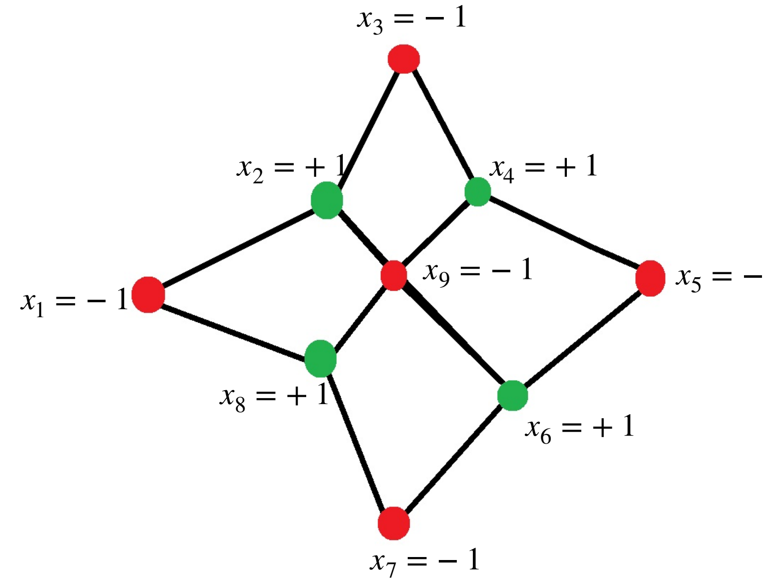
$\mathbf{x_i}$ is a random variable with $\{-1, +1\}$ outcomes



$$\mathbb{E}\Big(\sum_{ij} \mathbf{x}_i \mathbf{x}_j\Big)$$



Deterministic:



$$\sum_{ij} x_i x_j$$

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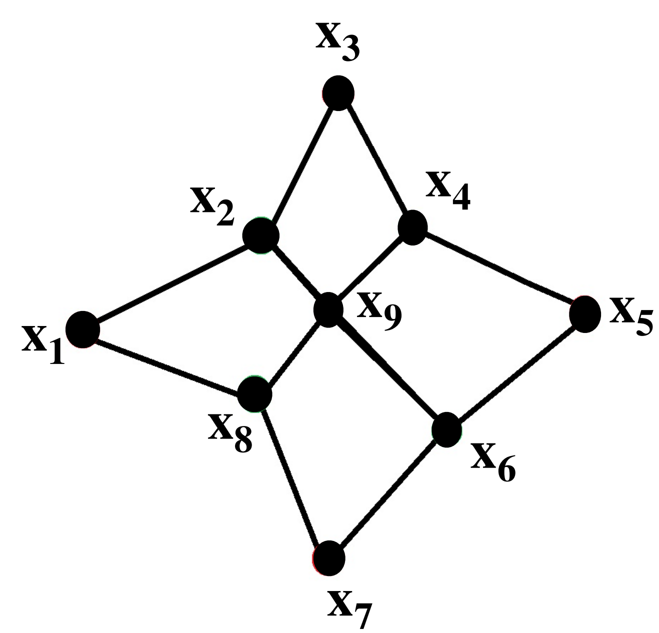
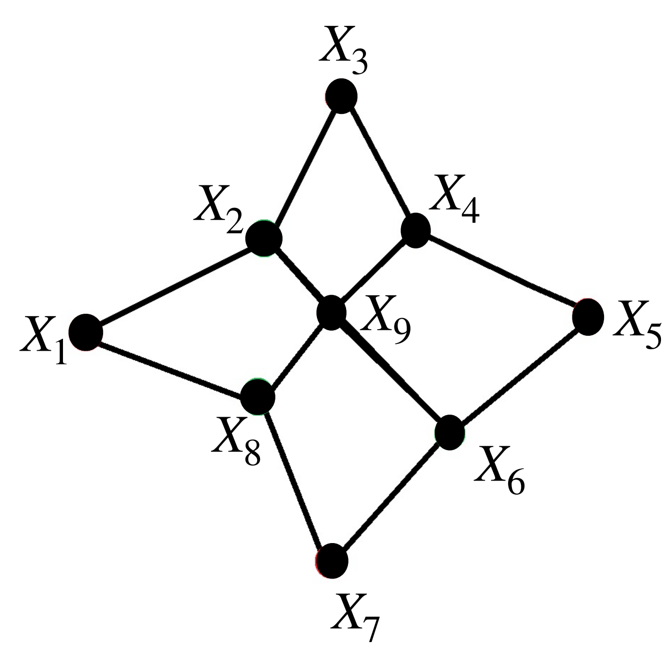


$$\mathbb{E}\Big(\sum_{ij} \mathbf{x}_i \mathbf{x}_j\Big)$$

Operator:

X_i is a unitary with $\{-1, +1\}$ eigenvalues



$$\mathrm{tr}\Big(\sum_{ij} X_i X_j\Big)$$

X_i is a unitary with $\{-1, +1\}$ eigenvalues in $\mathbb{C}^{d \times d}$

X_i is a binary observable

X_i is a unitary with $\{-1,+1\}$ eigenvalues in $\mathbb{C}^{d\times d}$

X_i is a binary observable

$$\mathrm{tr}(X_iX_j) = \langle\psi|X_iX_j\otimes I|\psi\rangle$$

where $|\psi\rangle = \frac{1}{\sqrt{d}}\sum_{t=0}^{d-1}|t\rangle\otimes|t\rangle$ is the MES

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$$\mathrm{tr}(X_i X_j) = \langle\psi|X_i X_j \otimes I|\psi\rangle$$

$$= \langle\psi|X_i \otimes X_j^T|\psi\rangle$$

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$$= \langle \psi | X_i \otimes X_j^T | \psi \rangle$$

$$= \langle \psi | X_i \otimes X_j | \psi \rangle$$

Simultaneously measuring observables X_i and X_j on halves of MES

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Simultaneously measuring observables X_i and X_j on halves of MES

$$= \Pr(x_ix_j = 1) - \Pr(x_ix_j = -1)$$

and letting x_i and x_j denote the outcome of measurements

Quantum probability

X_i is a unitary with $\{-1, +1\}$ eigenvalues in $\mathbb{C}^{d \times d}$

X_i is a binary observable

$$\mathrm{tr}(X_i X_j) = \langle \psi | X_i X_j \otimes I | \psi \rangle$$

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Quantum probability

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X_i is a binary observable

$$\text{tr}(X_i X_j) = \Pr(x_i x_j = 1) - \Pr(x_i x_j = -1)$$

Similarly in probability theory

$\mathbf{x}_i$ is a random variable with outcomes $\{-1, +1\}$

$\mathbf{x}_i$ is a binary random variable

Quantum probability

X_i is a unitary with $\{-1, +1\}$ eigenvalues in $\mathbb{C}^{d\times d}$

X_i is a binary observable

$$\mathrm{tr}(X_i X_j) = \Pr(x_i x_j = 1) - \Pr(x_i x_j = -1)$$

Similarly in probability theory

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$\mathbf{x}_i$ is a binary random variable

$$\mathbb{E}(\mathbf{x}_i \mathbf{x}_j) = \Pr(\mathbf{x}_i \mathbf{x}_j = +1) - \Pr(\mathbf{x}_i \mathbf{x}_j = -1)$$

Quantum probability

X_i is a unitary with $\{-1, +1\}$ eigenvalues in $\mathbb{C}^{d\times d}$

X_i is a binary observable

$$\mathrm{tr}(X_i X_j) = \Pr(x_i x_j = 1) - \Pr(x_i x_j = -1)$$

Similarly in probability theory

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$\mathbf{x}_i$ is a binary random variable

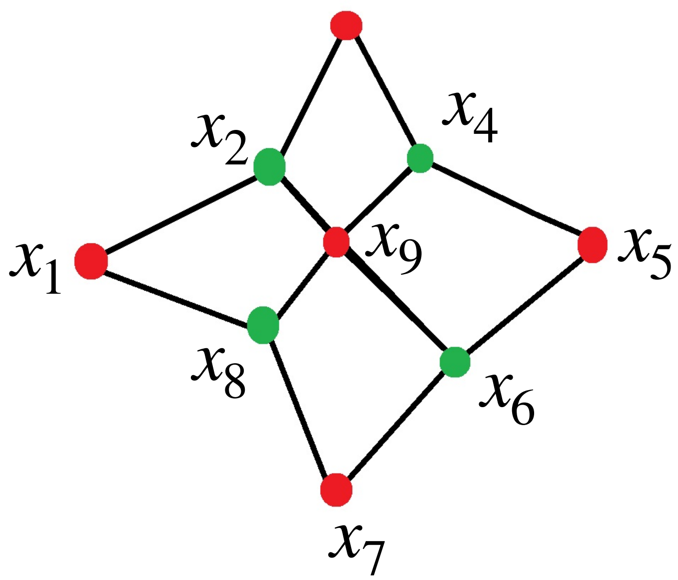
$$\mathbb{E}(\mathbf{x}_i \mathbf{x}_j) = \Pr(\mathbf{x}_i \mathbf{x}_j = +1) - (\mathbf{x}_i \mathbf{x}_j = -1)$$

Operator CSPs

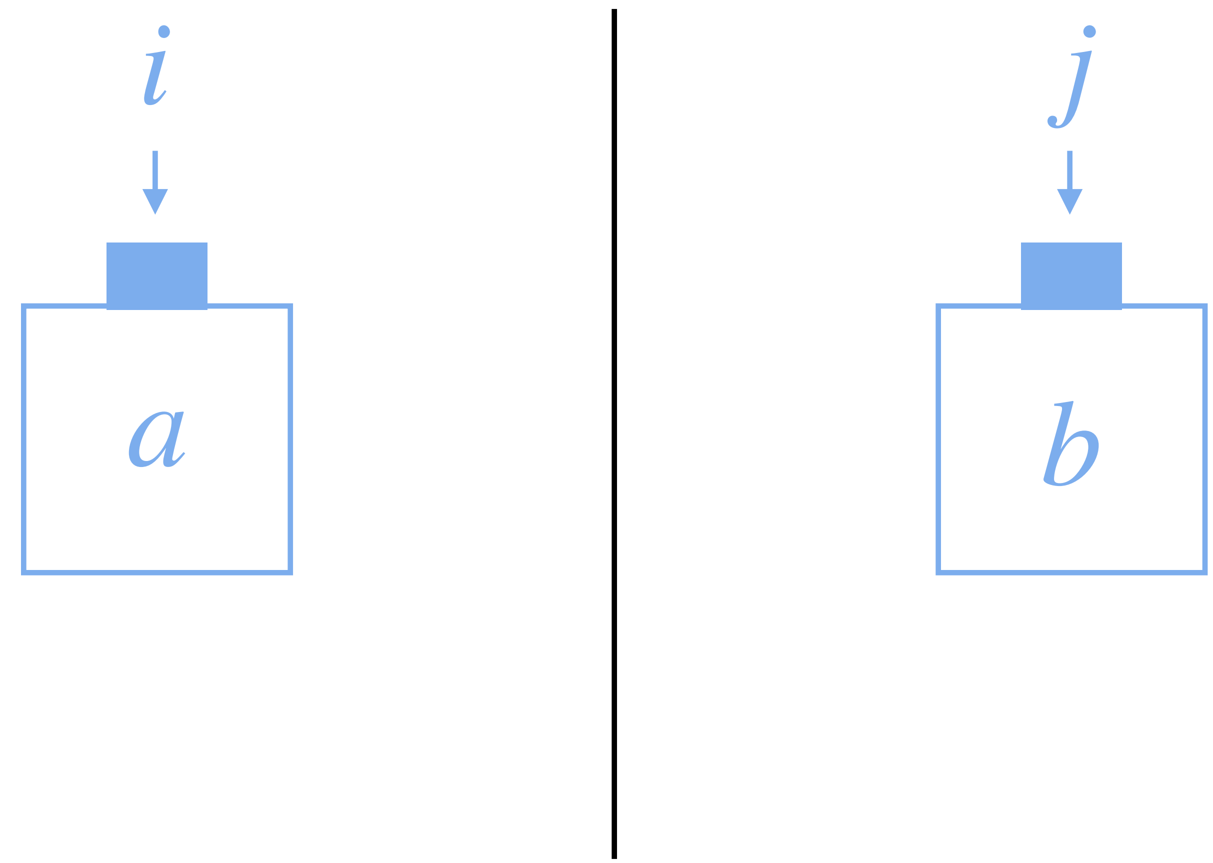
can be formulated as

entangled nonlocal games

Correlations, nonlocal games, Bell inequalities, …



$$\sum_{i,j} x_i x_j$$



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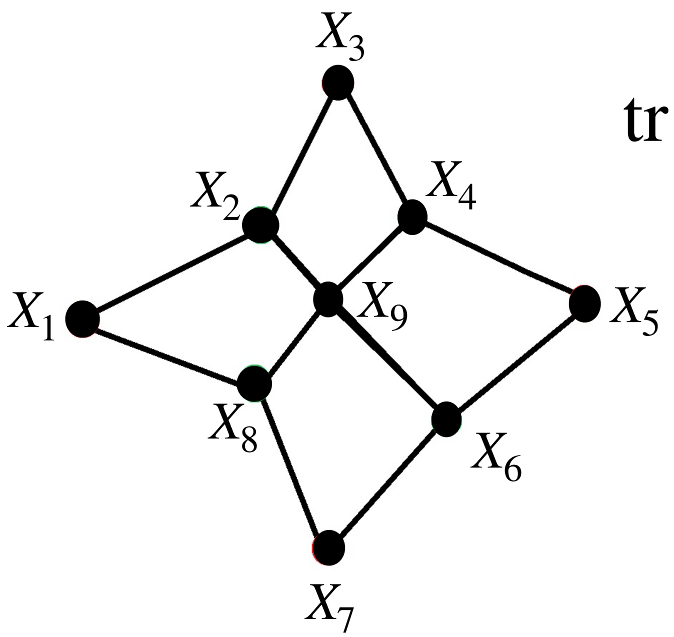
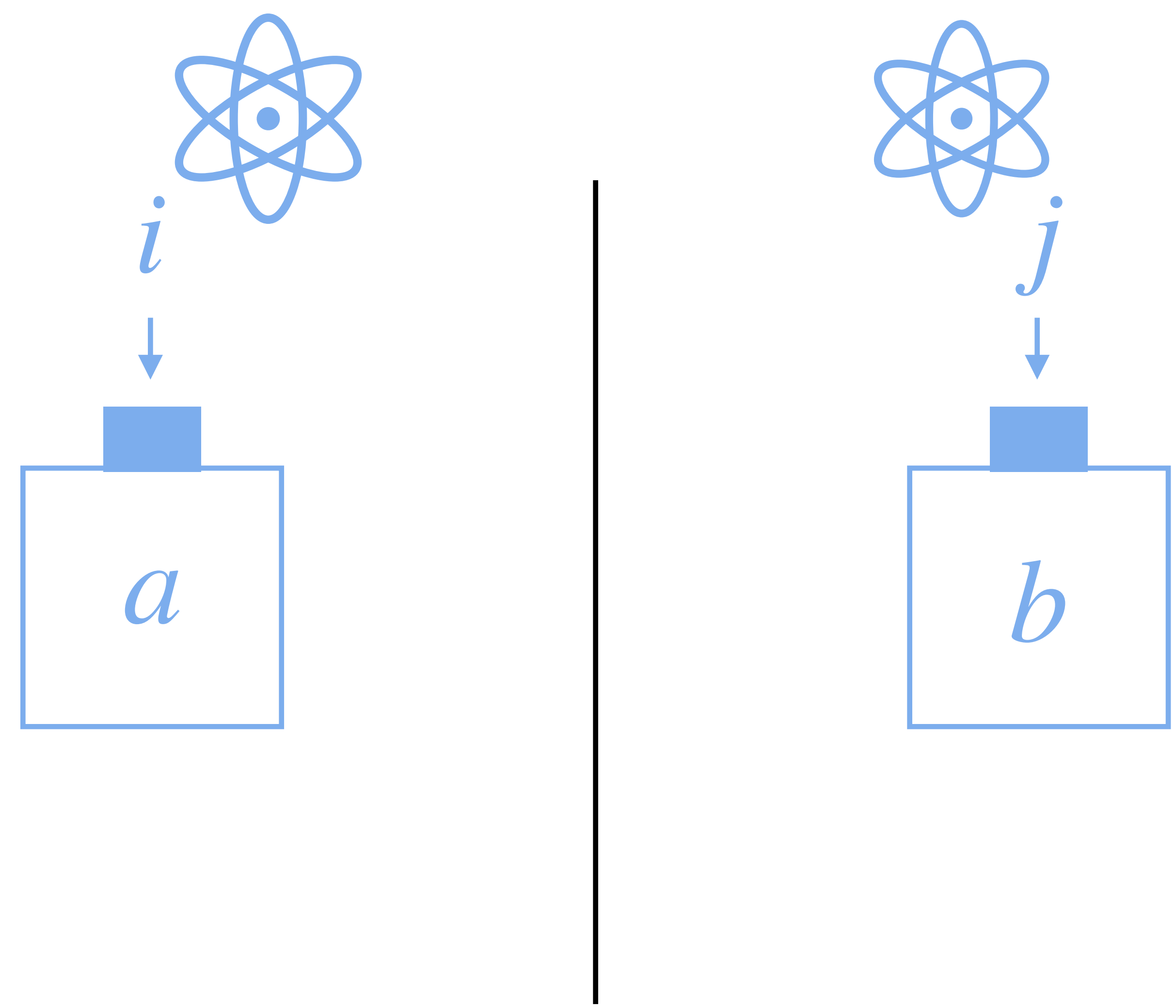
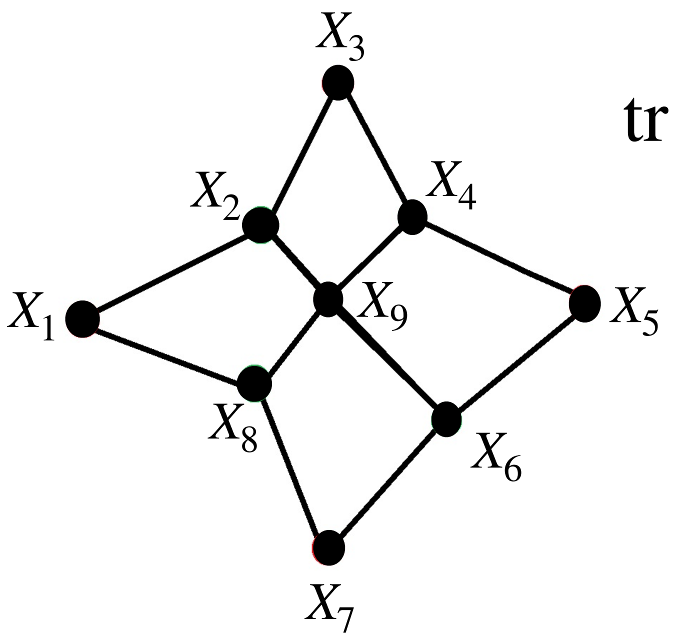


$$\mathrm{tr}\Big(\sum_{ij} X_i X_j\Big)$$



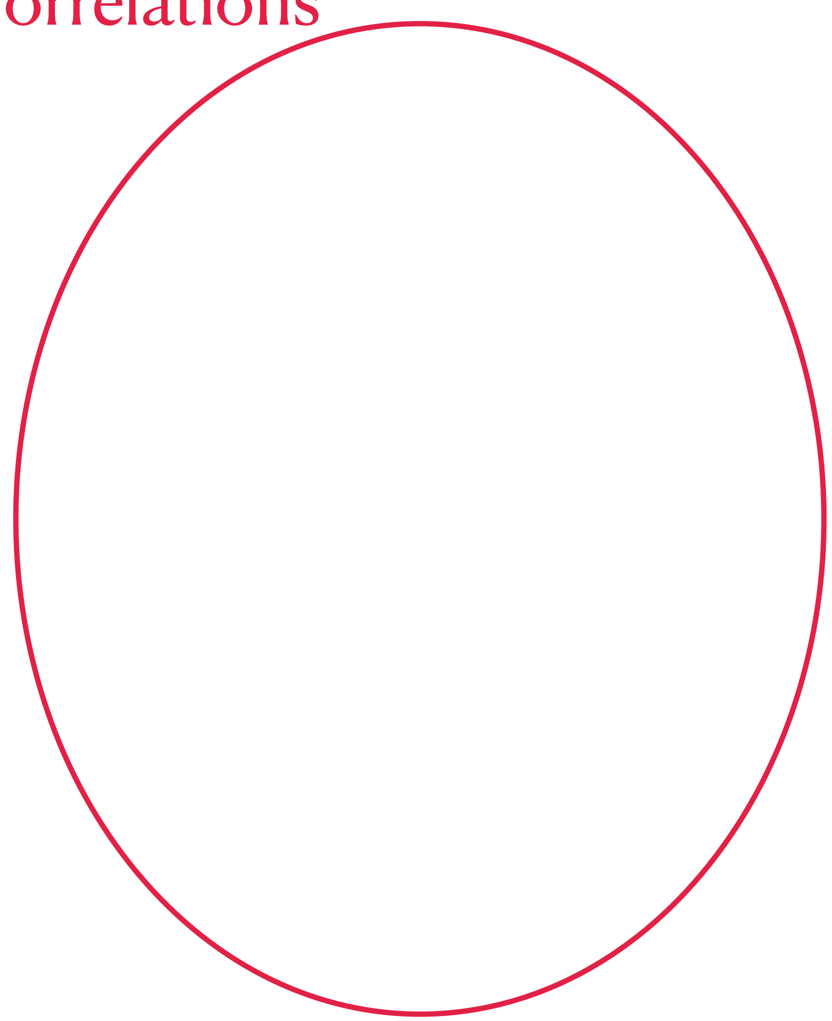
$$\Pr_{i,j}(a,b) = \mathrm{tr}\Big(\frac{I + aX_i}{2}\frac{I + bX_j}{2}\Big)$$

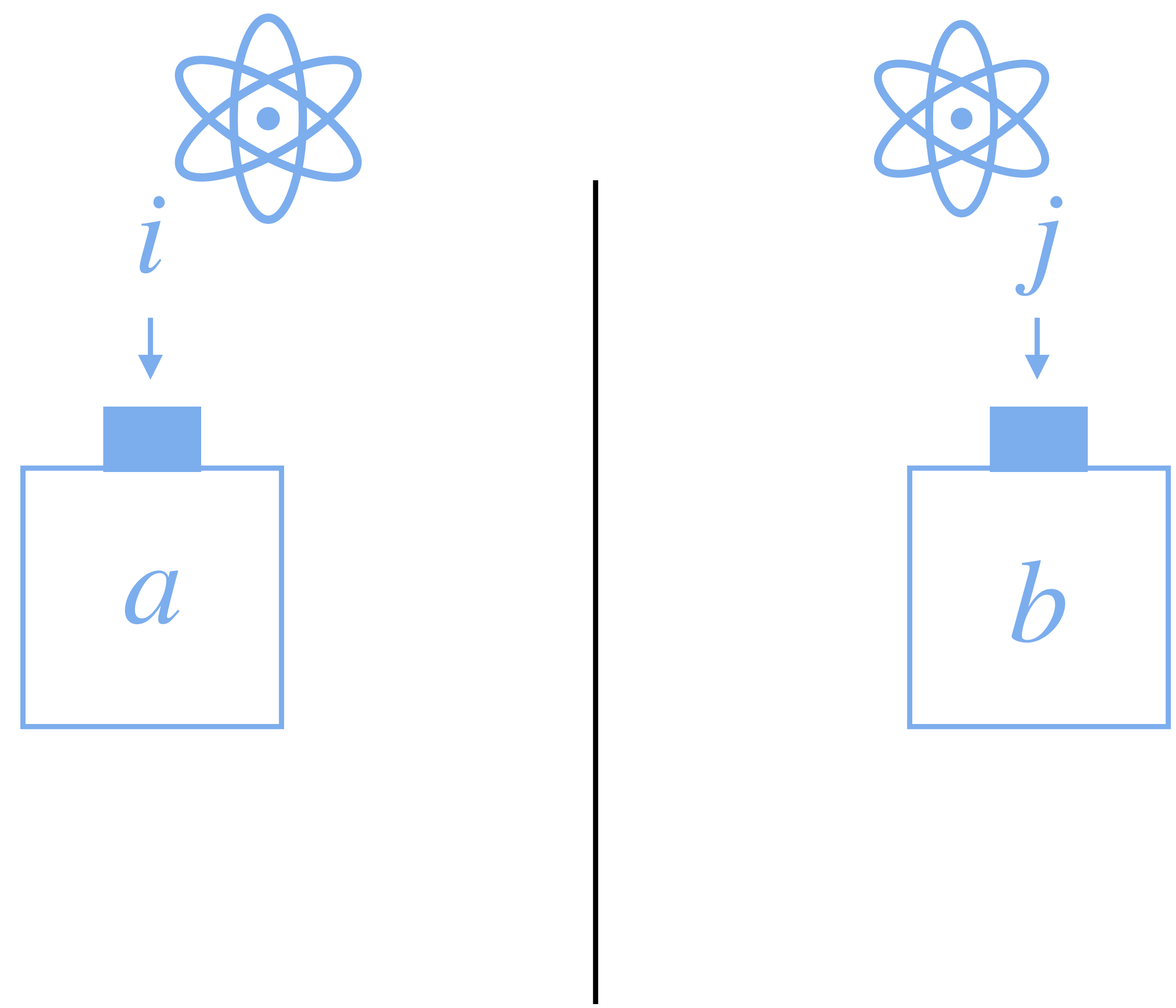
Correlations, nonlocal games, Bell inequalities, ...



$$\mathrm{tr}\Big(\sum_{ij} X_i X_j\Big)$$

Quantum Correlations





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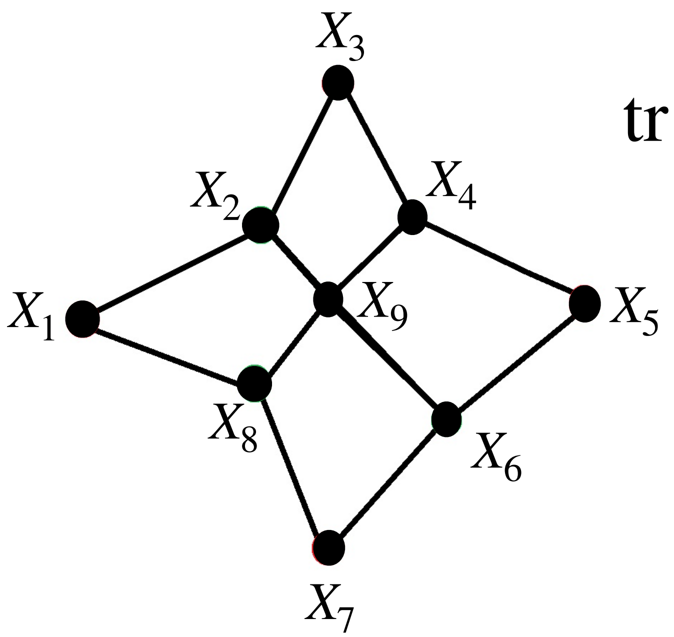

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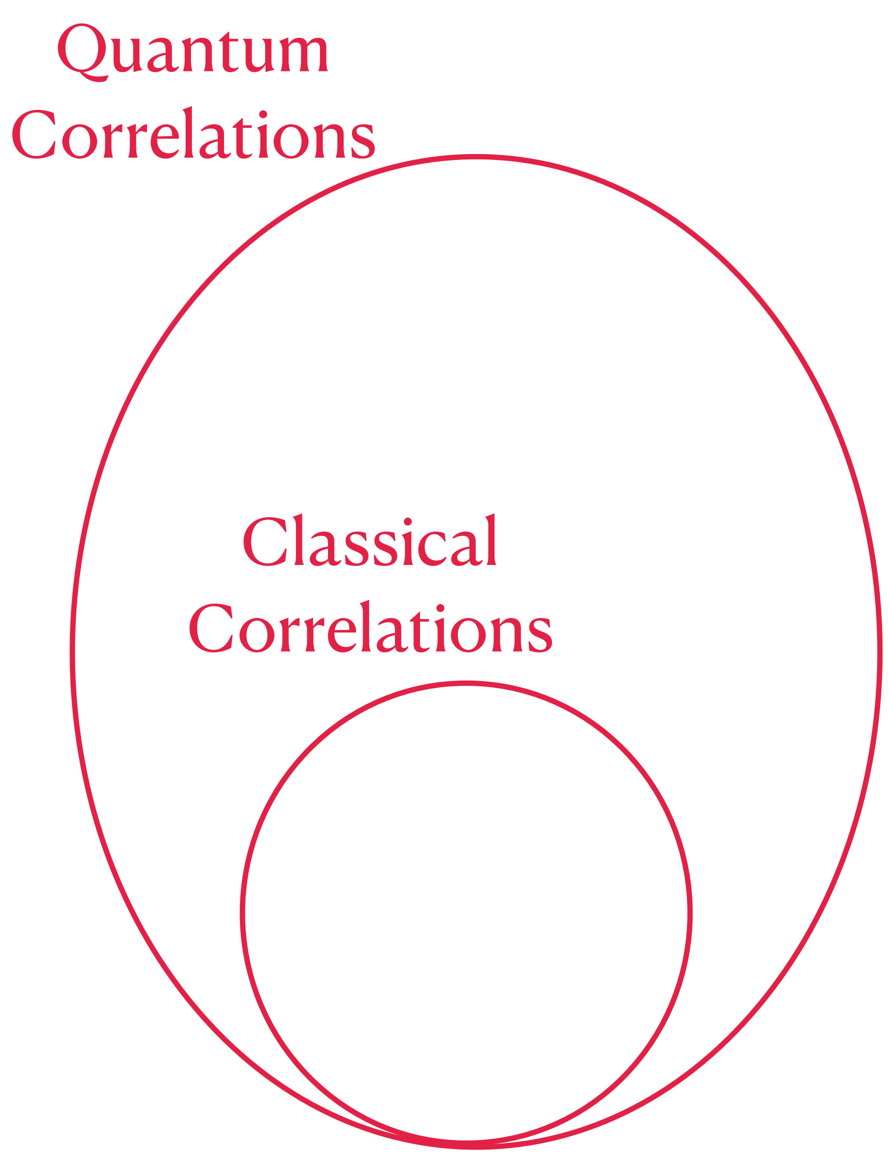
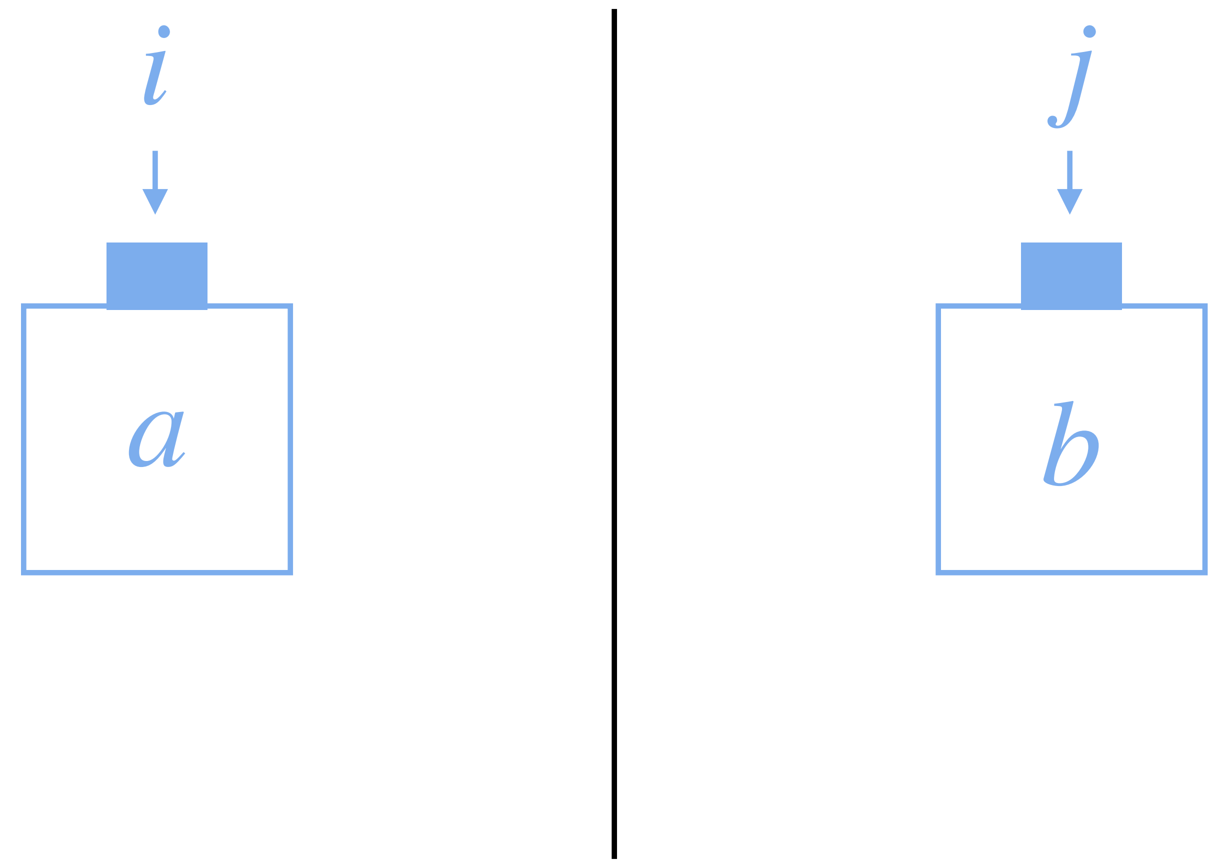
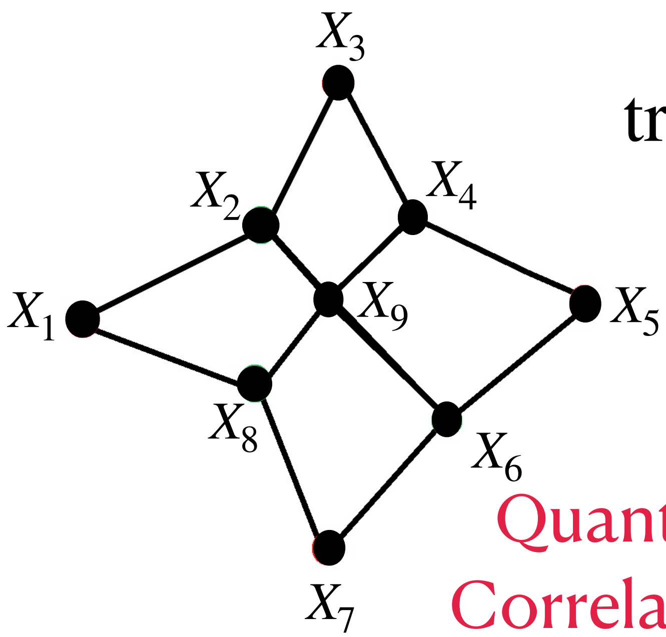


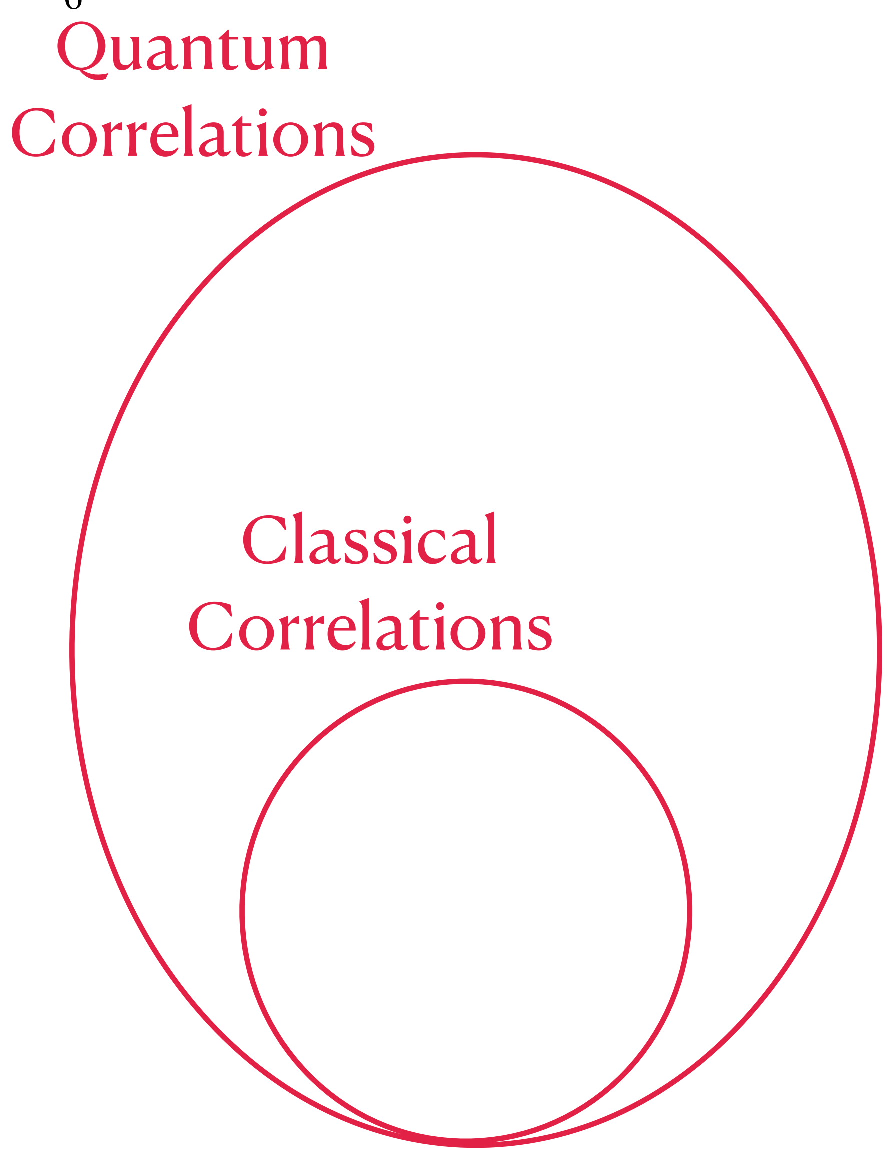


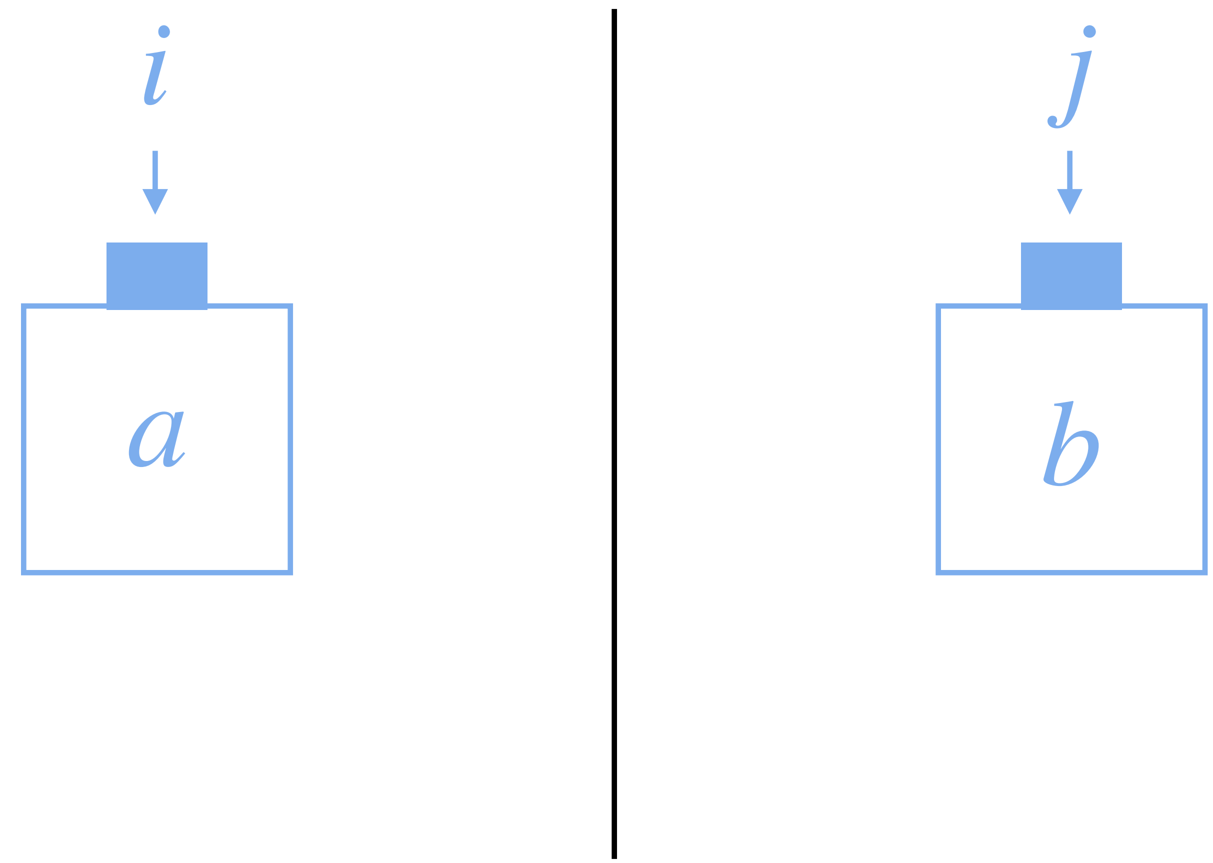
Correlations, nonlocal games, Bell inequalities, ...



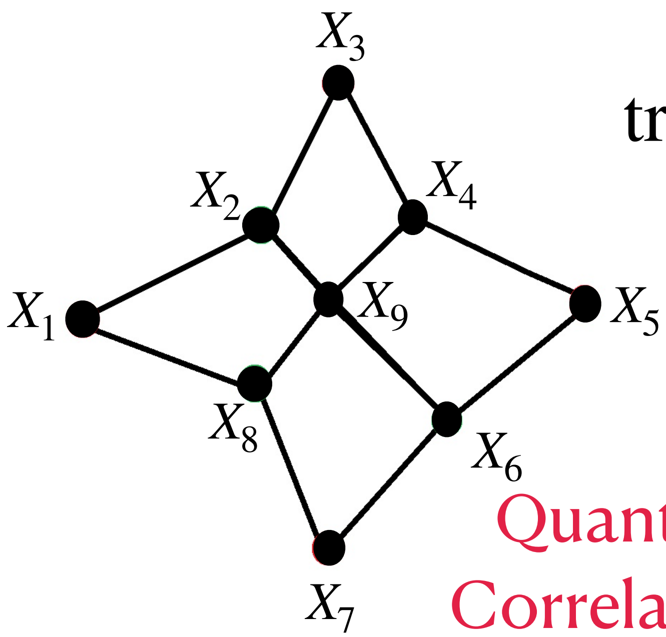
$$\mathrm{tr}\Big(\sum_{ij} X_i X_j\Big)$$

$\mathbb{R}^{4n^2}$





Correlations, nonlocal games, Bell inequalities, ...



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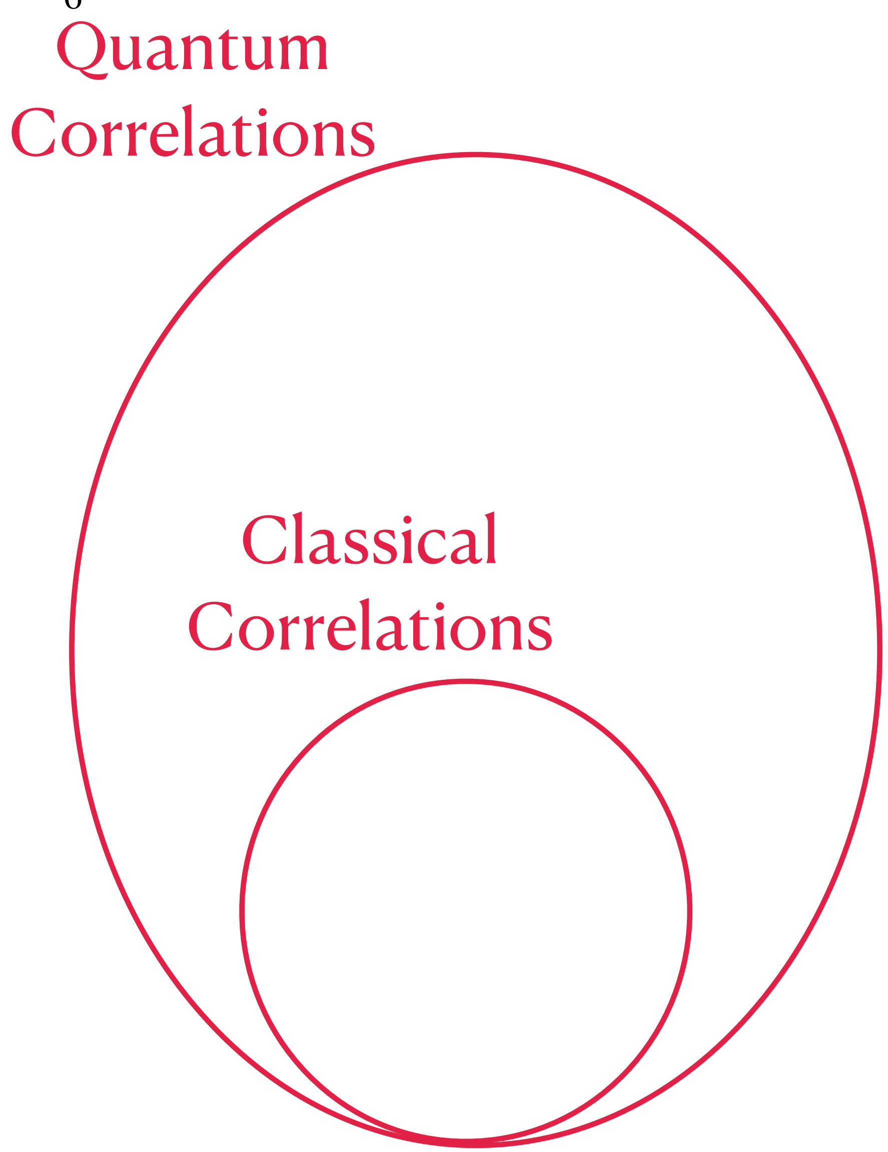
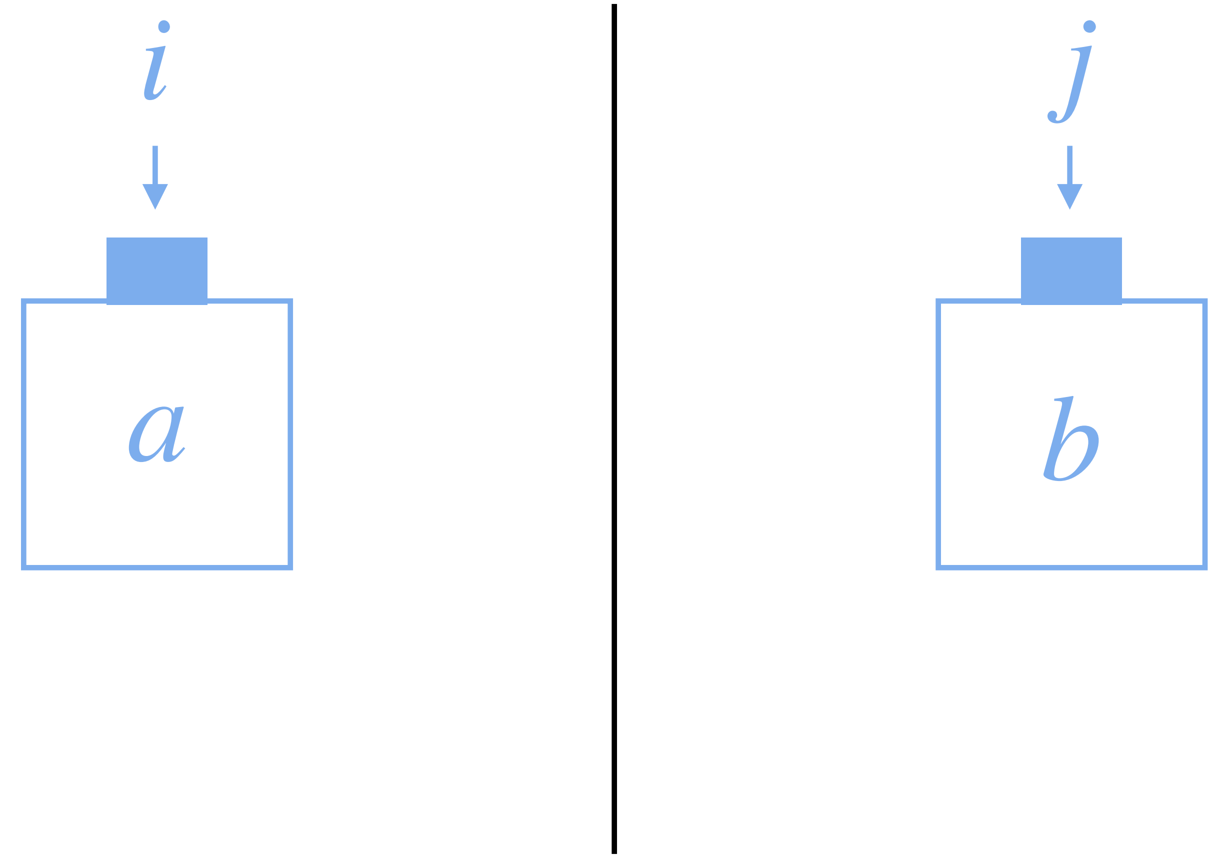
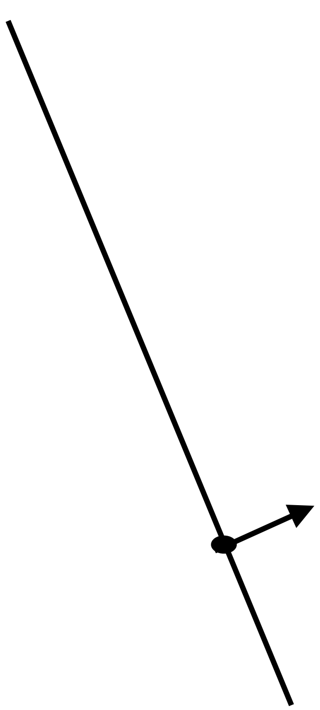
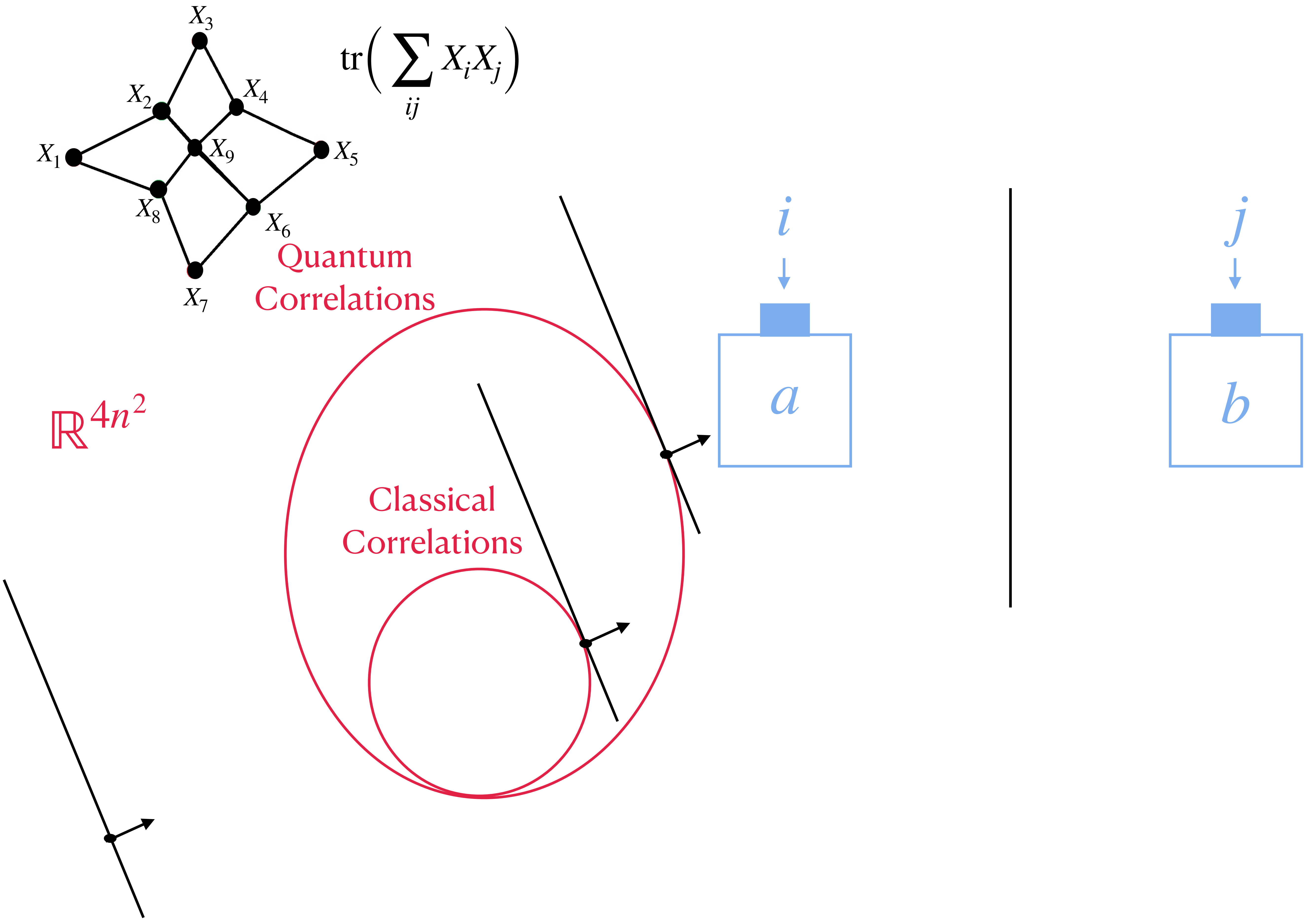






Correlations, nonlocal games, Bell inequalities, ...



for the rest of the talk all we care is …

CSP: polynomial optimization over $\mathbb{C}$

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i^2 = 1$

$x_i \in \mathbb{C}$

CSP: polynomial optimization over $\mathbb{C}$

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i^2 = 1$

$x_i \in \mathbb{C}$

OP-CSP: polynomial optimization over matrix algebras $\mathbb{C}^{d\times d}$

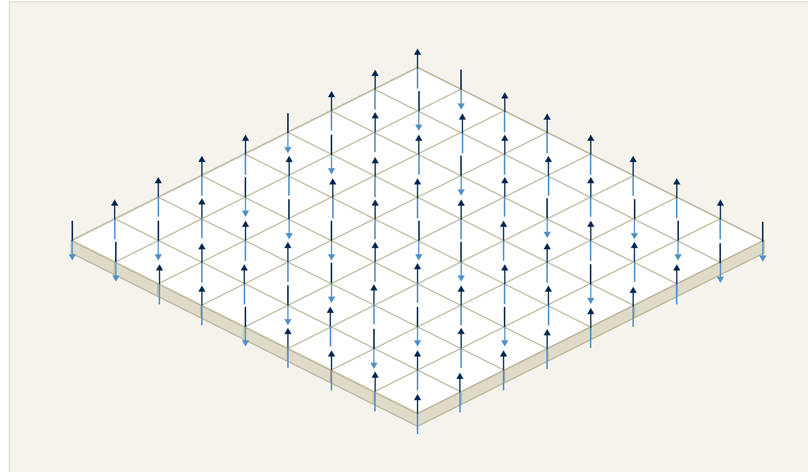
Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: $X_i^2 = X_i^* X_i = 1$

$X_i \in \mathbb{C}^{d\times d}$

$d \in \mathbb{N}$

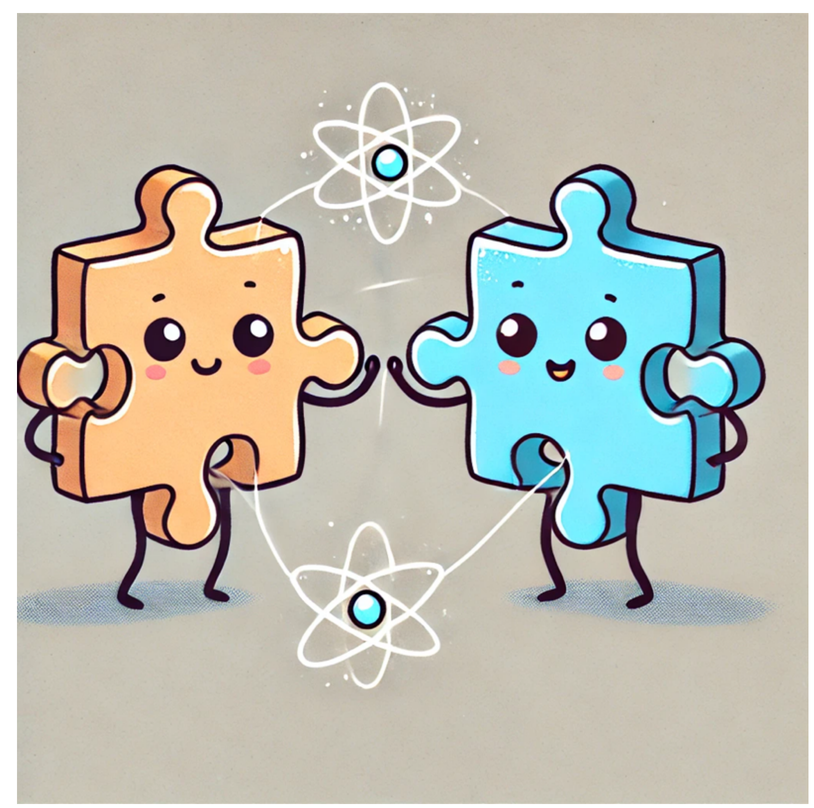
Approximation algorithms for constraint satisfaction problems



Constraint satisfaction

A rich extension

Operator constraint satisfaction



The algebraic nature of our tools

fits

the algebraic nature of CSPs and OP-CSPs

The **algebraic nature** of our tools

fits

the **algebraic nature** of CSPs and OP-CSPs

(sum-check protocol, low-degree testing, Fourier analysis on the hypercube)

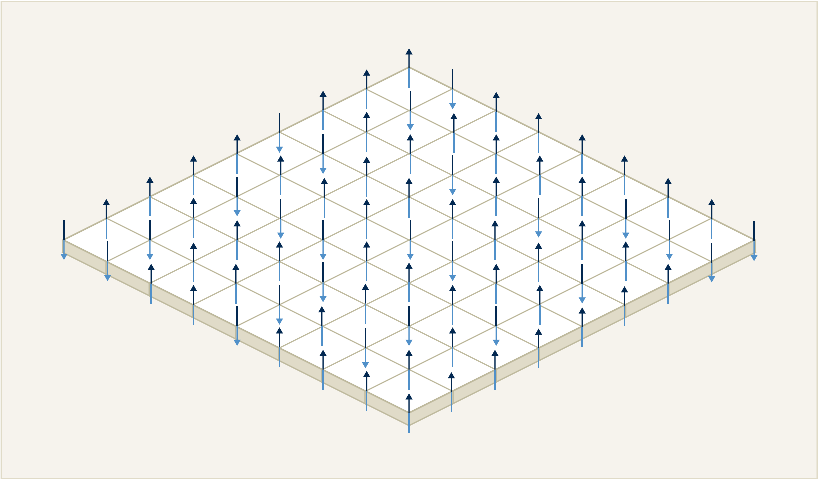
A classical theorem involving NP-hard and CSP

A classical theorem involving NP-hard and CSP

becomes

A theorem that involves RE-hard and OP-CSP

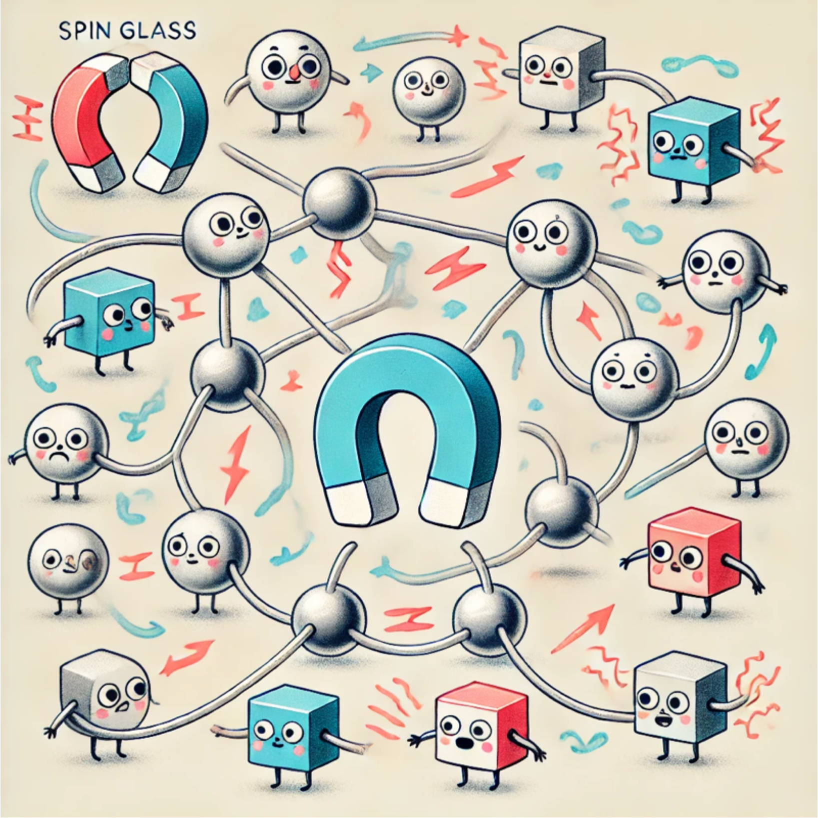
CSPs in classical and quantum worlds



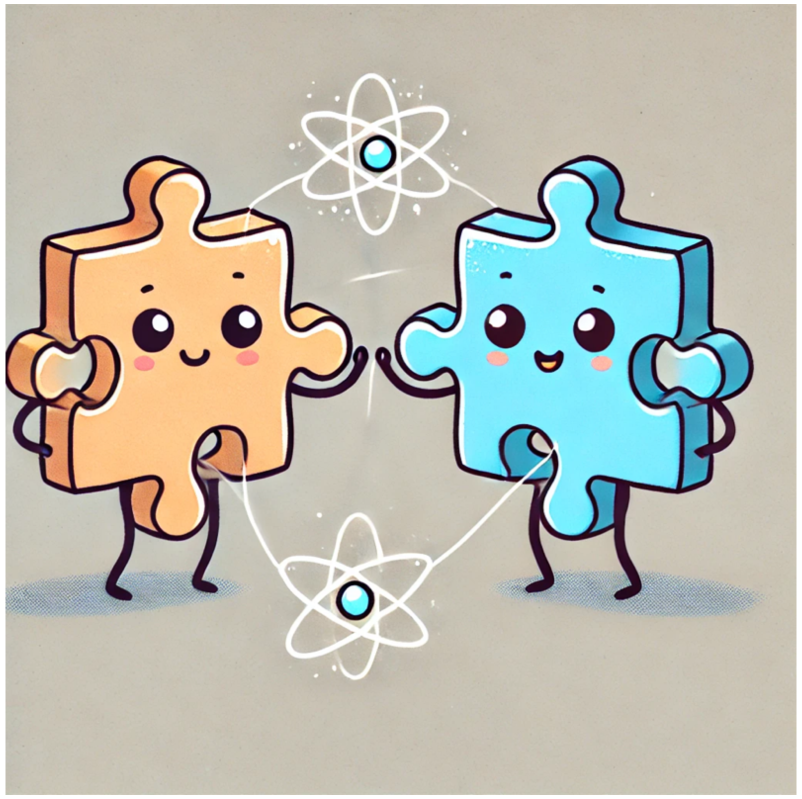
CSPs

A rich extension

Local Hamiltonians



Operator CSPs



Other Quantum CSPs

Local Hamiltonians

Operator Ising model

Minimize $\sum_{i,j} \operatorname{tr}(X_i X_j)$

Subject to: X_i is an observable

Operator Ising model → A feasible solution

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: X_i is an observable

$$X_i = \overbrace{I \otimes I \otimes \cdots \otimes \sigma^x}^{i} \otimes \cdots \otimes I = \sigma_i^x$$

Operator Ising model → A feasible solution

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: X_i is an observable

$$X_i = \overbrace{I \otimes I \otimes \cdots \otimes \sigma^x}^{i} \otimes \cdots \otimes I = \sigma_i^x$$

↓

Value of this solution

$$\sum_{i,j} \mathrm{tr}(\sigma_i^x \sigma_j^x)$$

Operator Ising model → A feasible solution

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

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Value of this solution

$$\sum_{i,j} \mathrm{tr}(\sigma_i^x \sigma_j^x)$$

Optimize over all states ← Value of this solution

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x | \psi \rangle$

Subject to: $|\psi\rangle$ is a state

Operator Ising model

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: X_i is an observable

Optimize over all states

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x | \psi \rangle$

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Operator Ising (optimize over operators)

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Optimize over all states

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Operator Ising (optimize over operators)

Minimize $\sum_{i,j} \mathrm{tr}(X_i X_j)$

Subject to: X_i is an observable

Optimize over all states equivalent to classical Ising

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x | \psi \rangle$

Subject to: $|\psi\rangle$ is a state

Minimize $\sum_{i,j} x_i x_j$

Subject to: $x_i^2 = 1$

Classical Ising model

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x | \psi \rangle$

Subject to: $|\psi\rangle$ is a state

Classical Ising model

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x | \psi \rangle$

Subject to: $|\psi\rangle$ is a state

Quantum Heisenberg model

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z | \psi \rangle$

Subject to: $|\psi\rangle$ is a state

Classical Ising model

Minimize $\sum_{i,j} \langle \psi | \sigma_i^x \sigma_j^x | \psi \rangle$

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Quantum Heisenberg model

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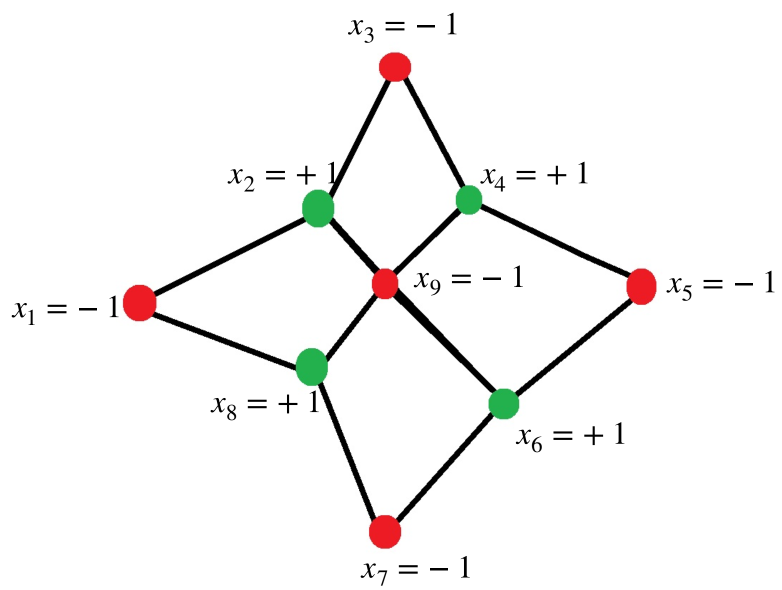
Subject to: $|\psi\rangle$ is a state

Local Hamiltonian terms: $\sigma_i^x \sigma_j^x + \sigma_i^y \sigma_j^y + \sigma_i^z \sigma_j^z$

A special case of the local Hamiltonian problem

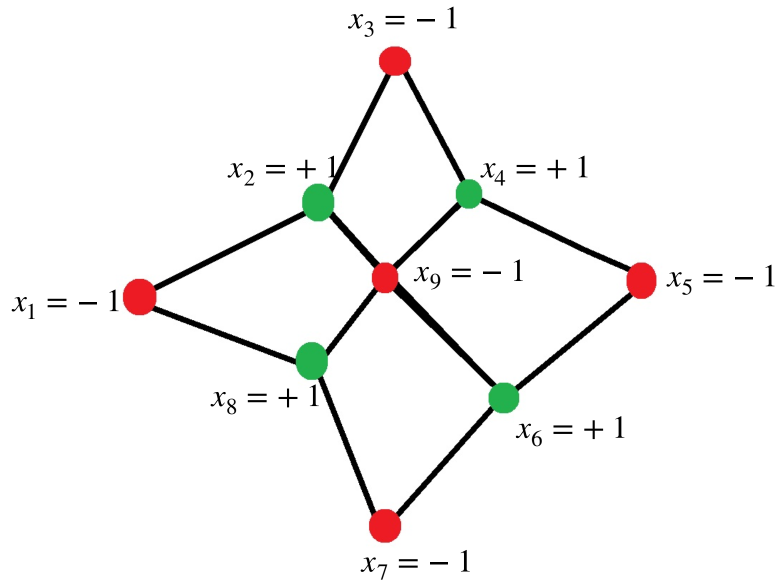


Variables:

$-1, +1$

Objective function:

$$x_1 x_2 + x_2 x_3 + \cdots$$



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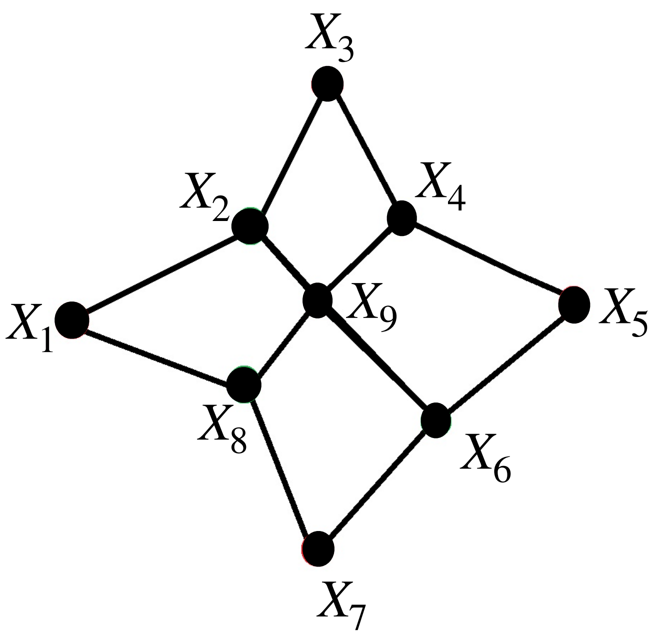
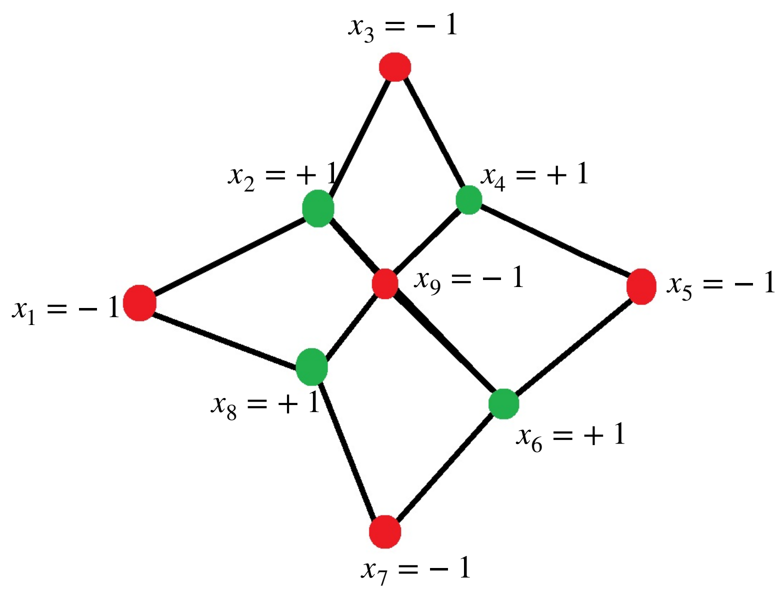


Observables

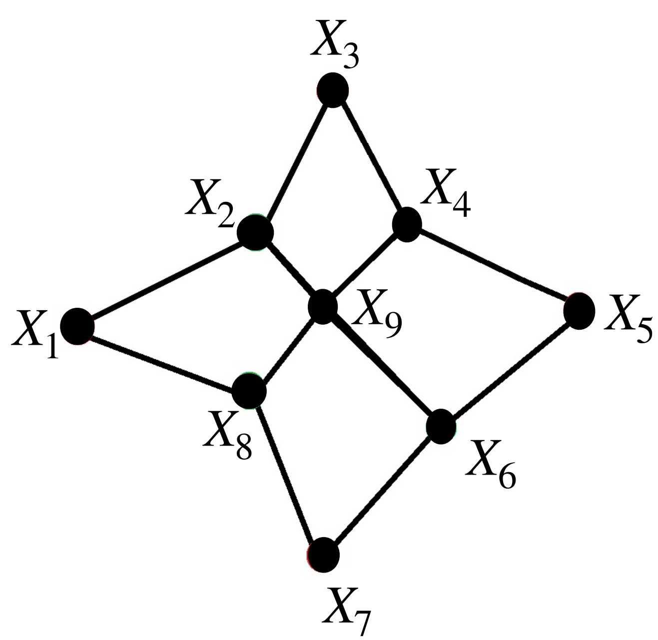
$$\mathrm{tr}\left(X_1 X_2 + X_2 X_3 + \cdots\right)$$

Variables: Objective function:



$-1, +1$

$$x_1 x_2 + x_2 x_3 + \cdots$$



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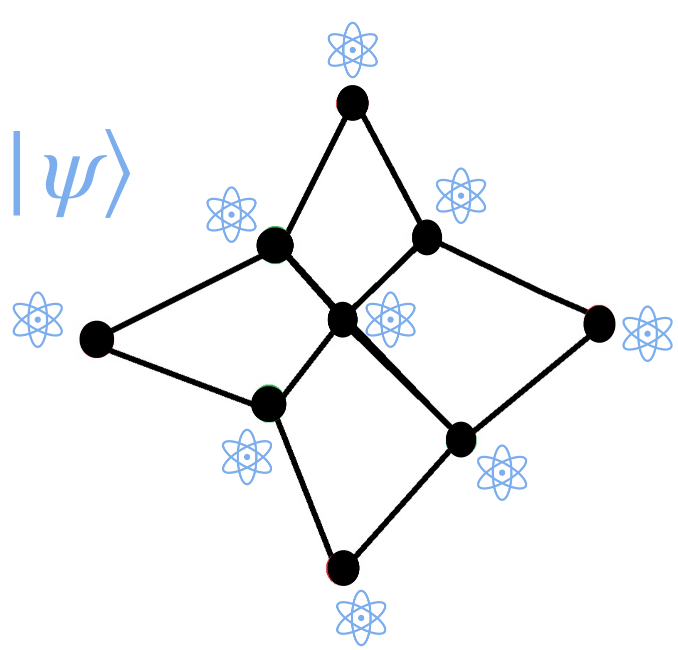
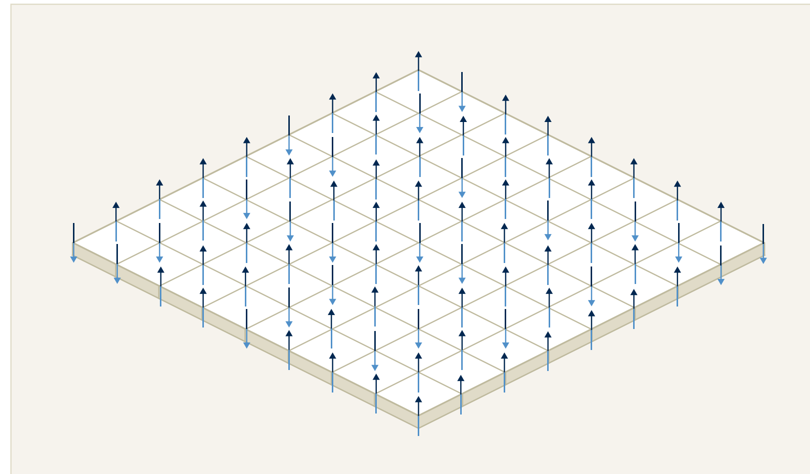


Qubits

$$\langle \psi | \sigma_1^x \sigma_2^x + \sigma_1^y \sigma_2^y + \sigma_1^z \sigma_2^z | \psi \rangle + \cdots$$

Not algebraic

Approximation algorithms for constraint satisfaction problems

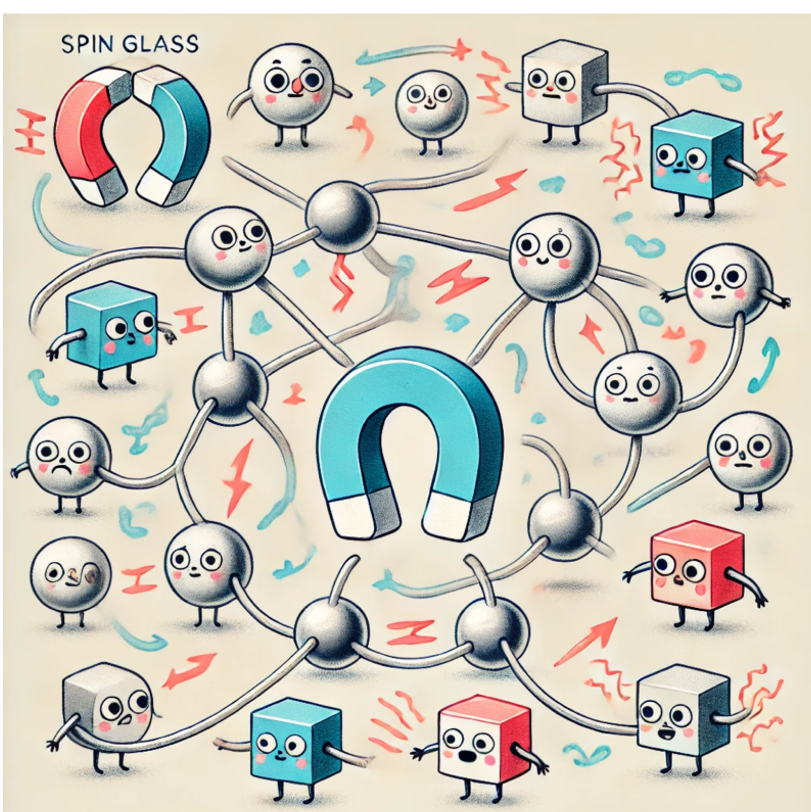


Classical CSPs

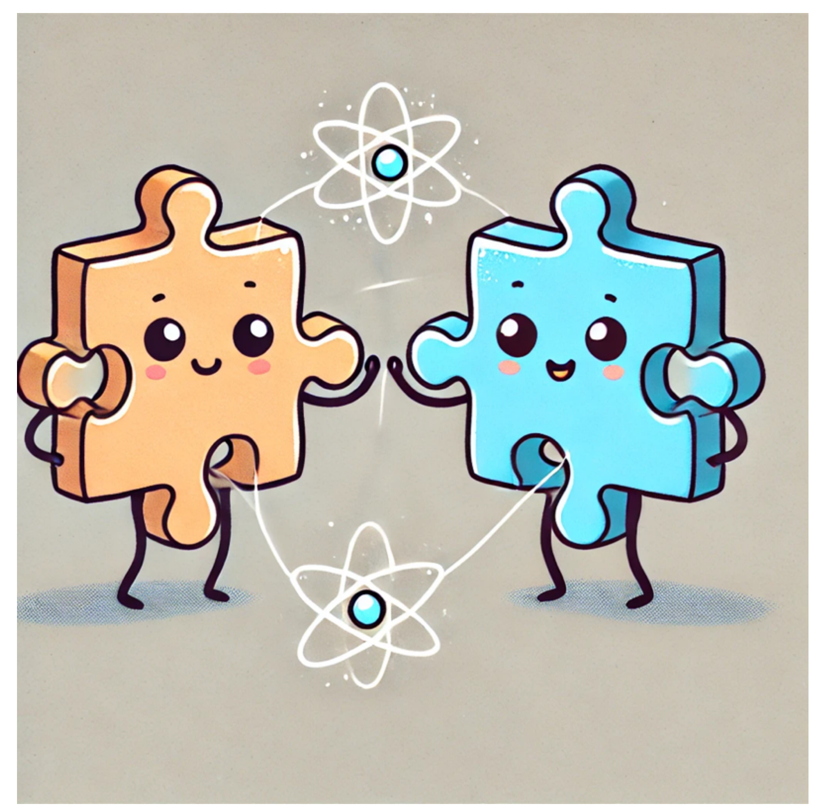
Does not extend!

A rich extension

Local Hamiltonians



OP-CSPs



CSPs: commutative algebras ☺️

OP-CSPs: matrix algebras 🙂

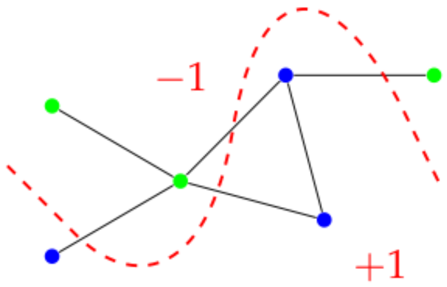
CSPs: commutative algebras ☺️

OP-CSPs: matrix algebras 🙂

Local Hamiltonians: not algebraic 🙁

Algorithms for OP-CSPs

How hard is OP-MaxCut?



MaxCut

max $\sum J_{ij} \; x_i x_j$

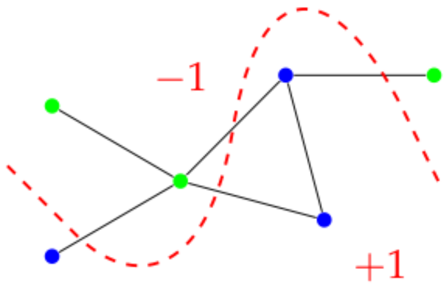
s.t. x_i is ± 1

OP-MaxCut

max $\sum J_{ij} \; tr(X_i X_j)$

s.t. X_i is an observable

How hard is OP-MaxCut?



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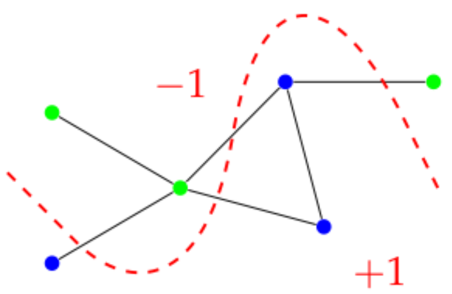
- Karp 1972: MaxCut is NP-Complete (is hard)

How hard is OP-MaxCut?



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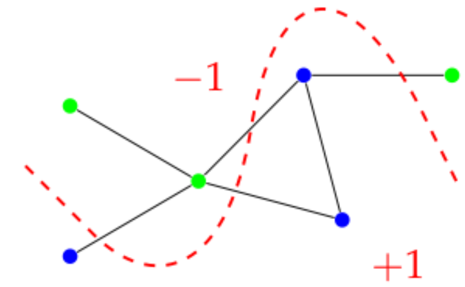

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- The best algorithm for MaxCut is SDP rounding by Goemans and Williamson

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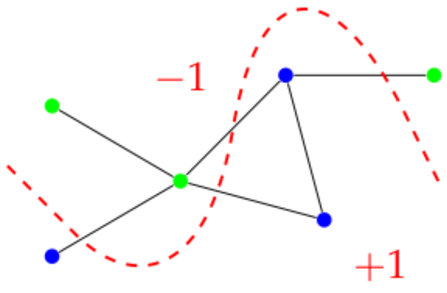

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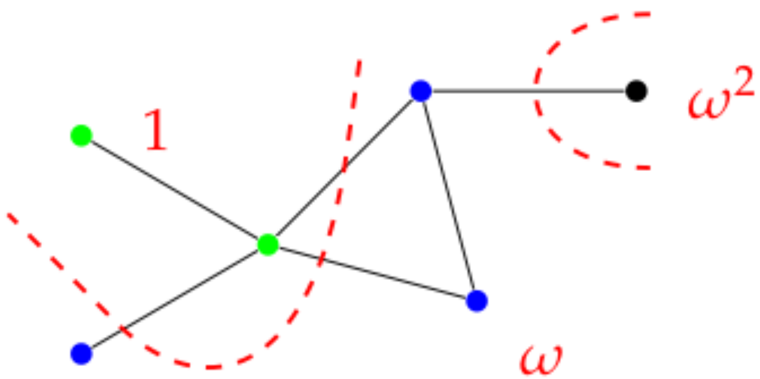
- The best algorithm for MaxCut is SDP rounding by Goemans and Williamson
- Tsirelson's algorithm is an operator generalization of Goemans and Willamson

Max-3-Cut



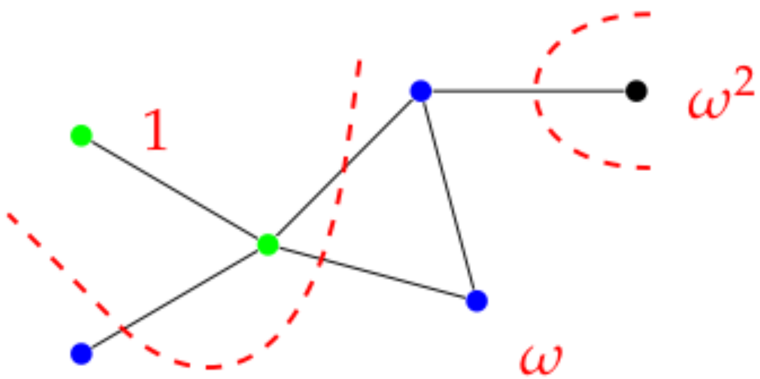
(a) Example of a partition of vertices into three subsets

$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{2 - x_i^* x_j - x_j^* x_i}{3}$$

$$\text{subject to:} \quad x_i \in \{1, \omega, \omega^2\},$$

(b) Max-3-Cut as a polynomial optimization

Max-3-Cut



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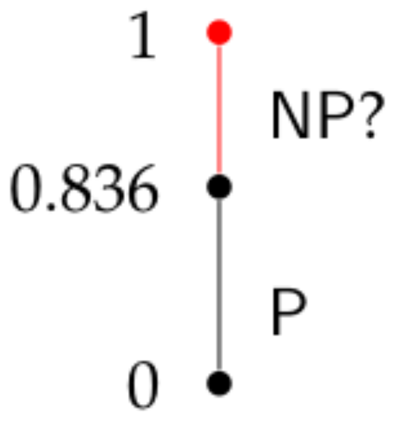
(b) Max-3-Cut as a polynomial optimization

OP-Max-3-Cut

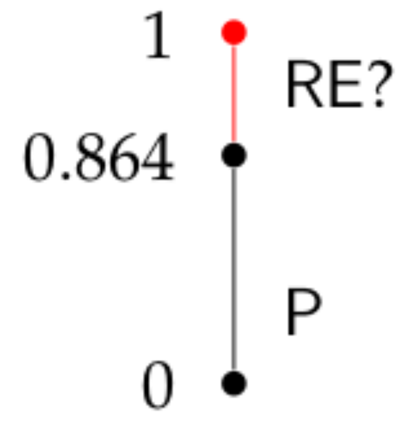
$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{2 - \langle X_i, X_j \rangle - \langle X_j, X_i \rangle}{3}$$

$$\text{subject to:} \quad X_i \text{ unitary with eigenvalues } 1, \omega, \omega^2.$$

Best algorithms for Max-3-Cut



(a) Max-3-Cut



(b) Noncommutative Max-3-Cut

Frieze and Jerrum

Culf, **M.**, Spirig

A classical theorem involving NP-hard and CSP

becomes

A theorem that involves RE-hard and OP-CSP

Hardness of Approximation for OP-CSPs

Hardness front: PCP theorem

- PCP theorem: Approximating Label-Cover is NP-hard
(Arora, Safra, Lund, Motwani, Sudan, Szegedy, Raz, Håstad)

- NC-PCP theorem (MIP*=RE): Approximating OP-Label-Cover is RE-hard
(Ji, Natarajan, Vidick, Wright, Yuen 2020)

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- NC-PCP theorem (MIP*=RE): Approximating OP-Label-Cover is RE-hard
(Ji, Natarajan, Vidick, Wright, Yuen 2020)

- Compare this with the situation for the Local Hamiltonian problem (LH):

Quantum PCP conjecture: Approximating Local Hamiltonian is QMA-hard

Hardness front: Unique games conjecture (UGC)

- Similarly UGC has an operator analogue

- Assuming UGC, approximating MaxCut to better than 0.878 is NP-hard (Khot, Kindler, Mossel, O'Donnell)

- Assuming Q-UGC, approximating Q-MaxCut to better than 0.878 is RE-hard (**M.**, Spirig)

*Q-MaxCut is a version of OP-MaxCut we did not discuss in the talk!

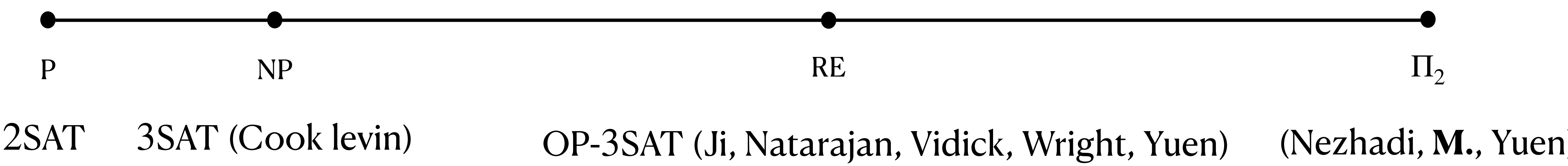
OP-CSPs and complexity classes

OP-CSPs are expressive

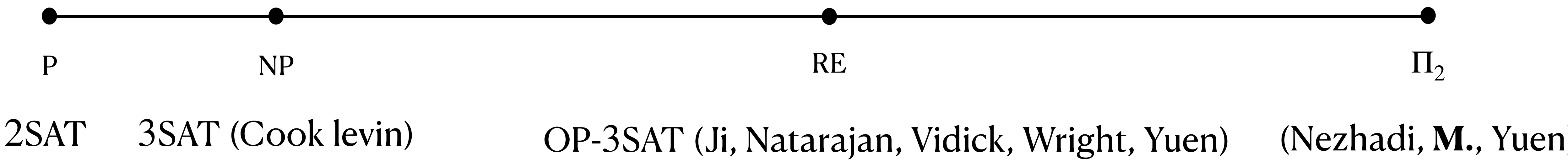
P NP RE

2SAT 3SAT (Cook levin) OP-3SAT (Ji, Natarajan, Vidick, Wright, Yuen)

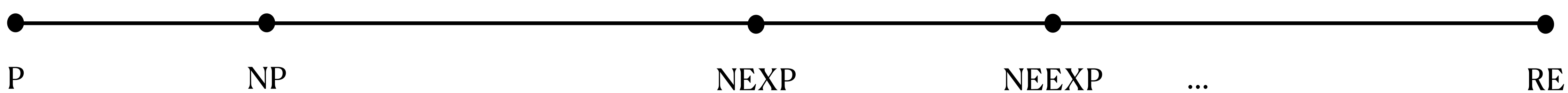
OP-CSPs are expressive



OP-CSPs are expressive



They also capture all the nondeterministic classes



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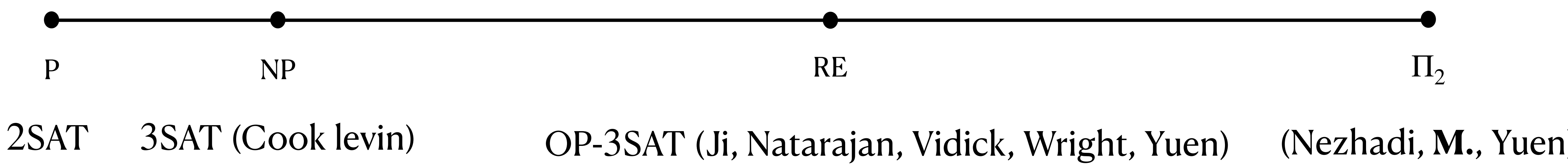

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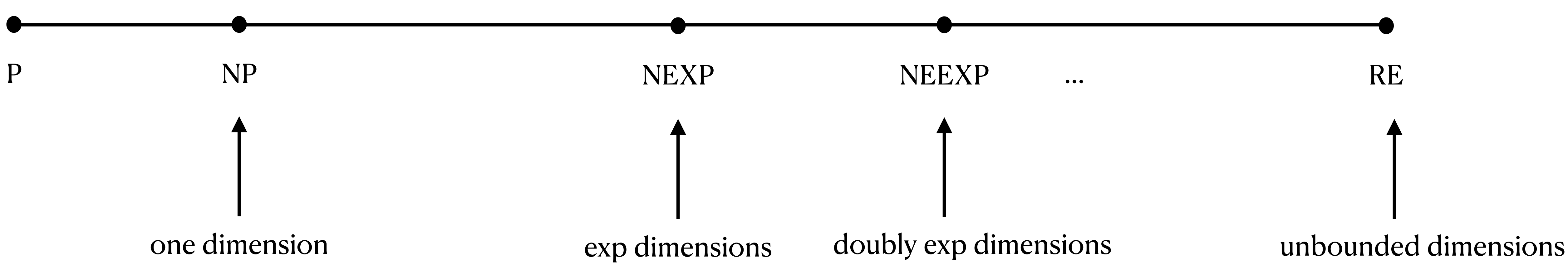
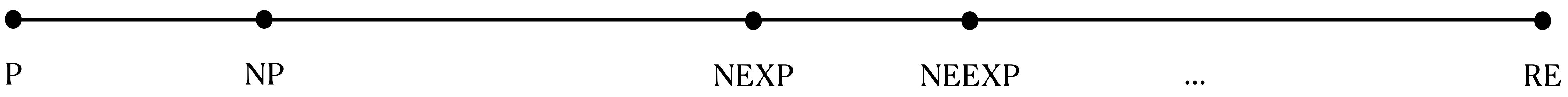
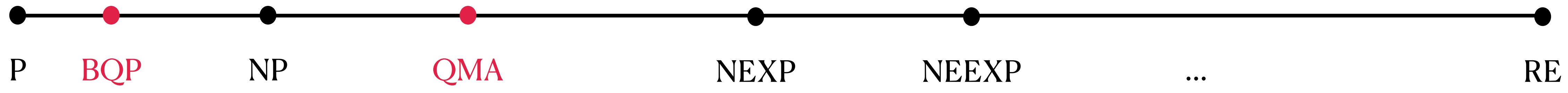


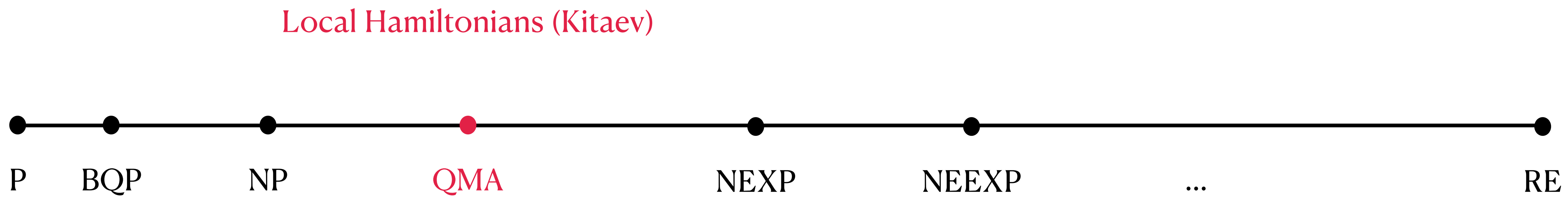
But they skip on quantum complexity classes!



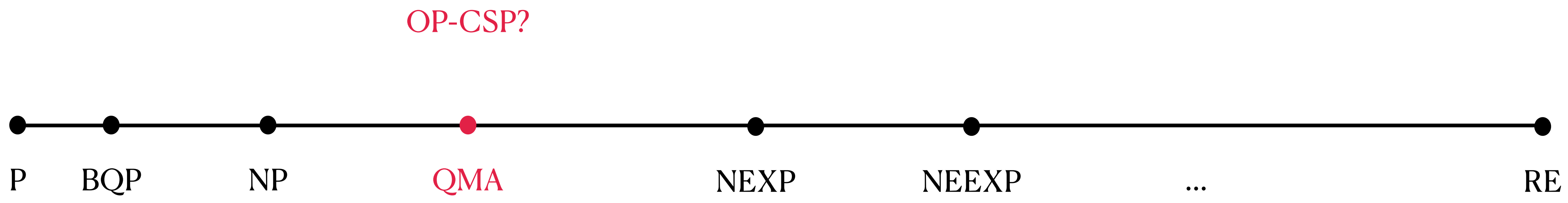
But they skip on quantum complexity classes



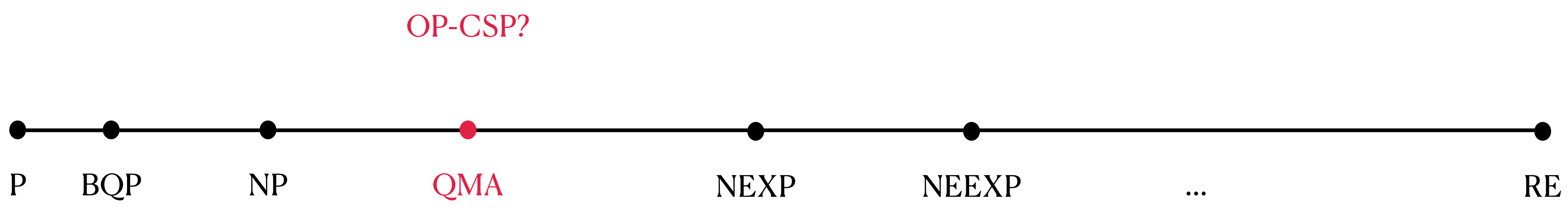
Local-Hamiltonian fills the gap



Open problem

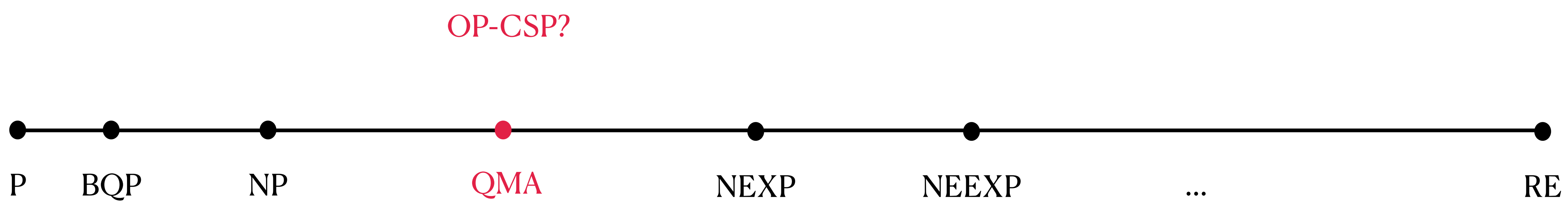


Open problem



- Restricting the dimension of observable => nondeterministic classes
- Requiring that the observables are efficiently implementable

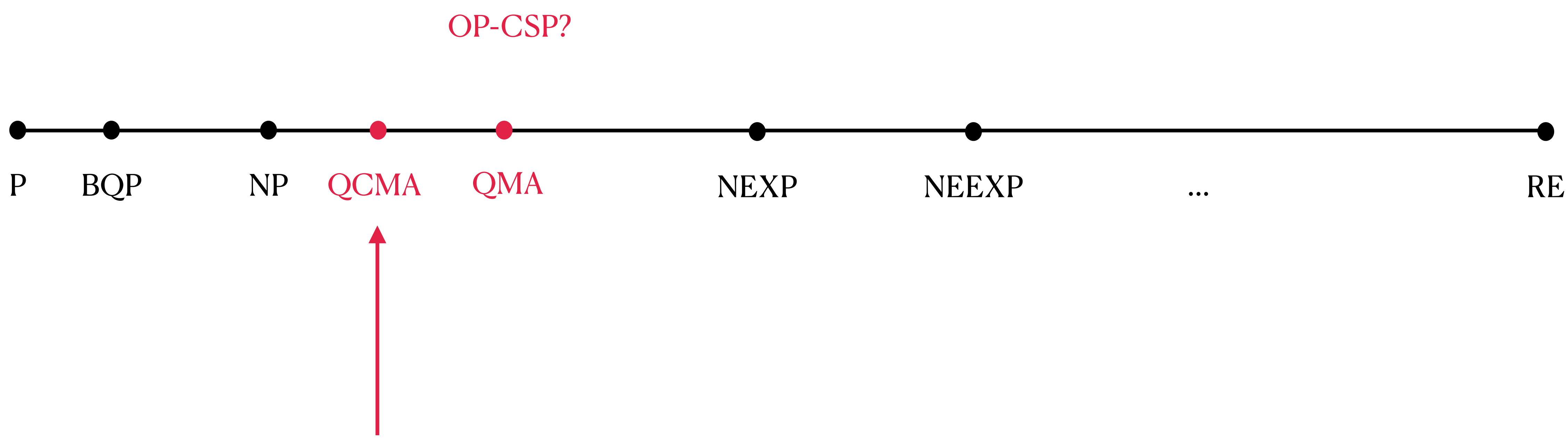
Open problem



$$\max \sum tr(X_i X_j)$$

s.t. X_i is an observable with an efficient circuit

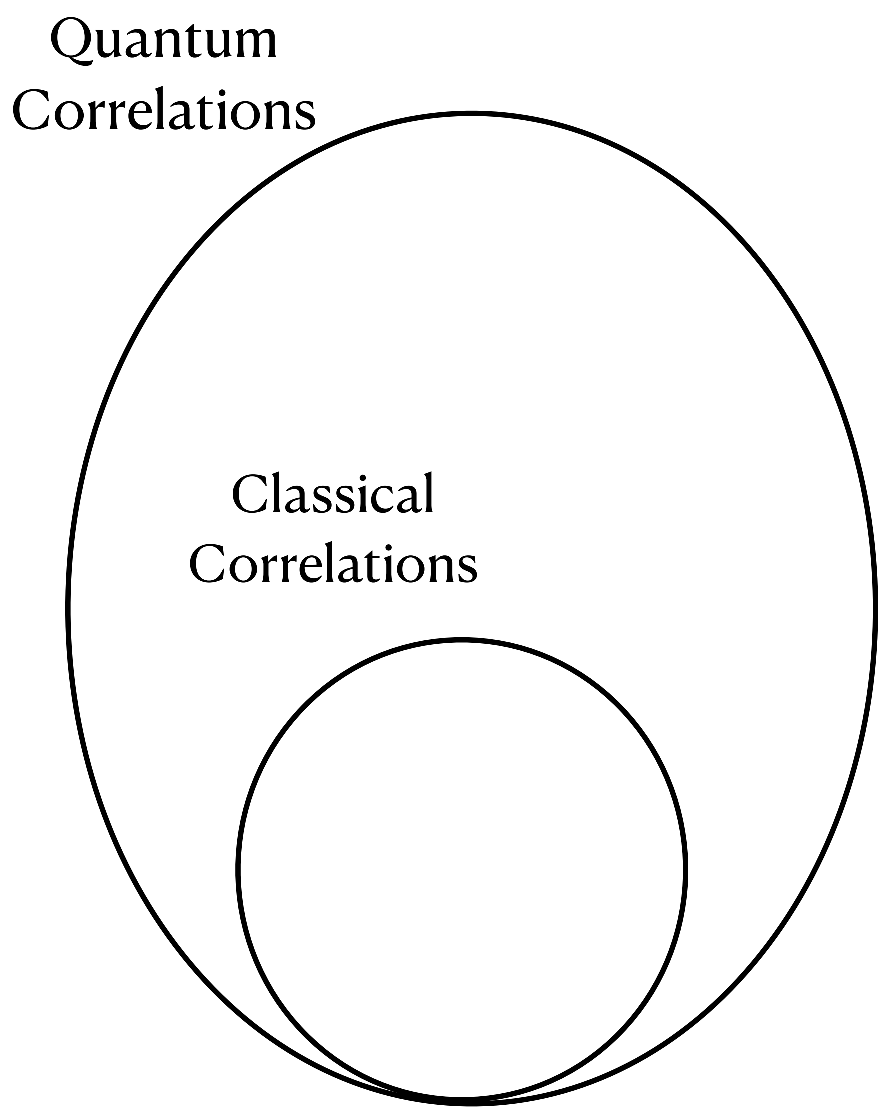
Open problem



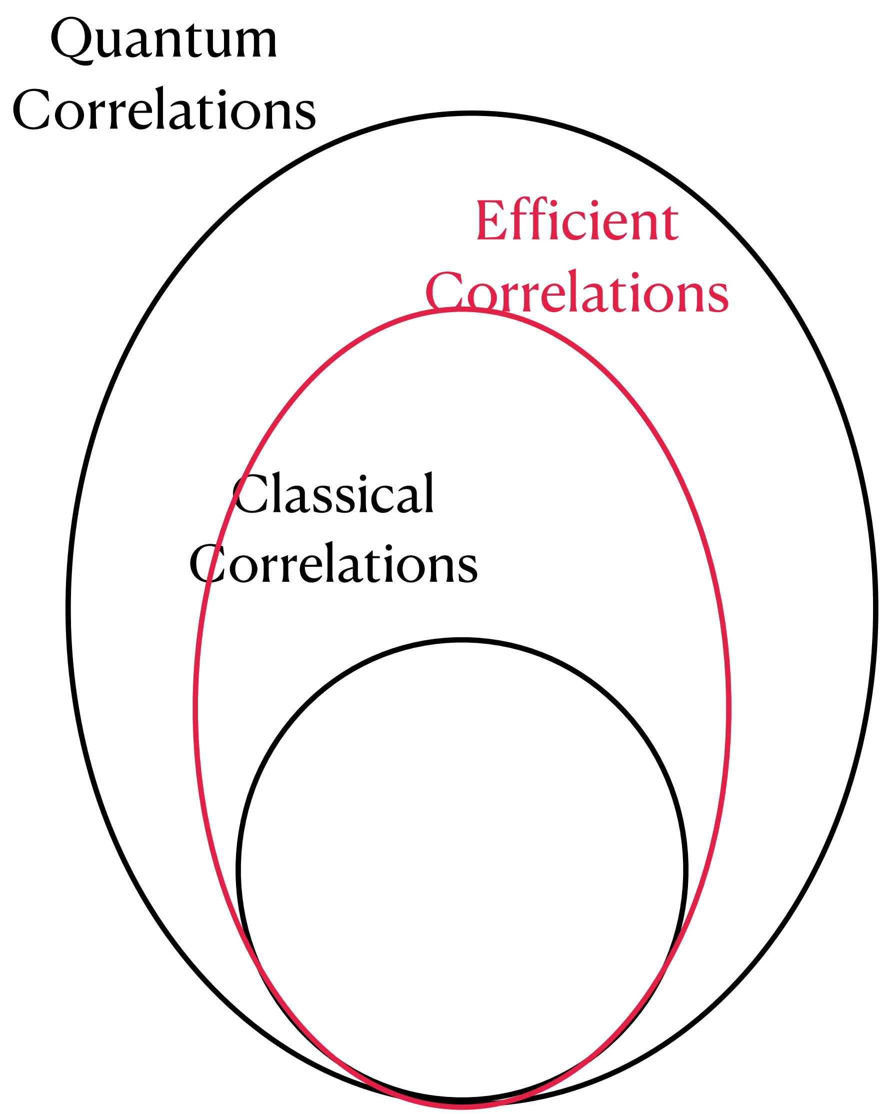
$$\max \sum tr(X_i X_j)$$

s.t. X_i is an observable with an efficient circuit

Set of correlations



Efficiently generated correlations



Summary

- Two generalization of CSPs in quantum information
    - Local Hamiltonians
    - OP-CSPs
- OP-CSPs share the algebraicity of classical CSPs
- We have been able to reach almost the same maturity in OP-CSPs
- Many of the CS tools applicable to CSPs are algebraic in nature
- For Local Hamiltonian we need to invent new tools
- But we may be able to understand QMA better
    - if we find an OP-CSP that captures it!

But why in Computer Science?

Magic Square

$x_{ij} \in \{+1, -1\}$

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

Perfect Operator Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$$\longrightarrow$$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

$$x_{ij} \in \{+1, -1\}$$

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x_{21}	x_{22}	x_{23}	$+1$
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$x_{ij} \in \{+1, -1\}$

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Binary alphabet $\{+1, -1\}$ in the classical case $\longrightarrow$ Binary observables

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$x_{ij} \in \{+1, -1\}$

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Binary alphabet $\{+1, -1\}$ in the classical case $\longrightarrow$ Binary observables

Binary observables: Unitary operators with $\{+1, -1\}$ eigenvalues

$$O^*O = O^2 = I$$

An operator CSP

$$X_{ij}^* X_{ij} = I$$

$$X_{ij}^2 = I$$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
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An operator CSP

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X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

When restricting to one dimension we recover the classical CSP

Because ± 1 are the only binary observables is one dimension

Perfect Operator Solution: algebraic structure

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Uniqueness of the perfect solution

$X_{ij}^* X_{ij} = I$

$X_{ij}^2 = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

$X_{11}X_{12} = X_{12}X_{11}, \quad X_{12}X_{21} = -X_{21}X_{12}, \quad \cdots$