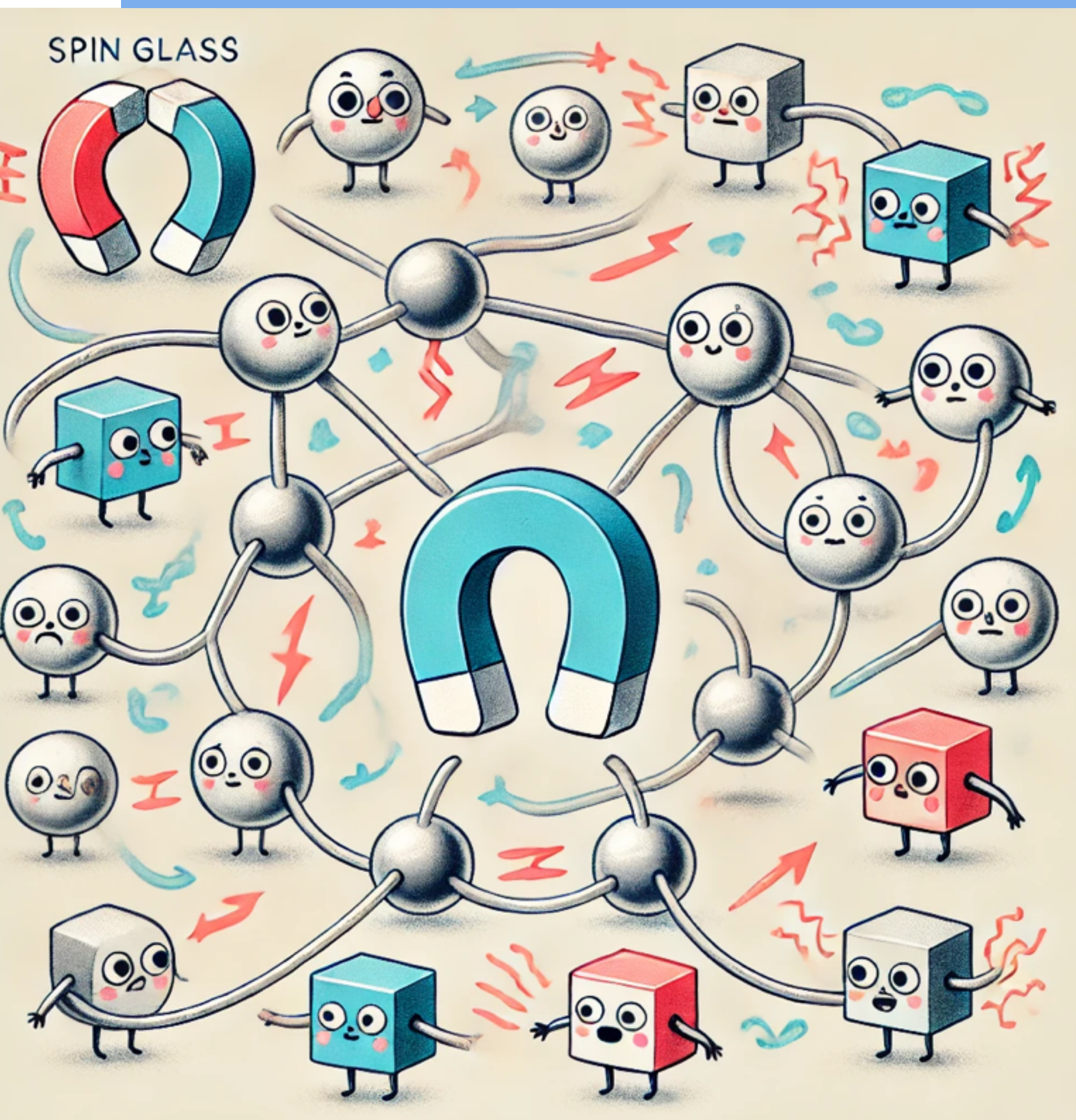
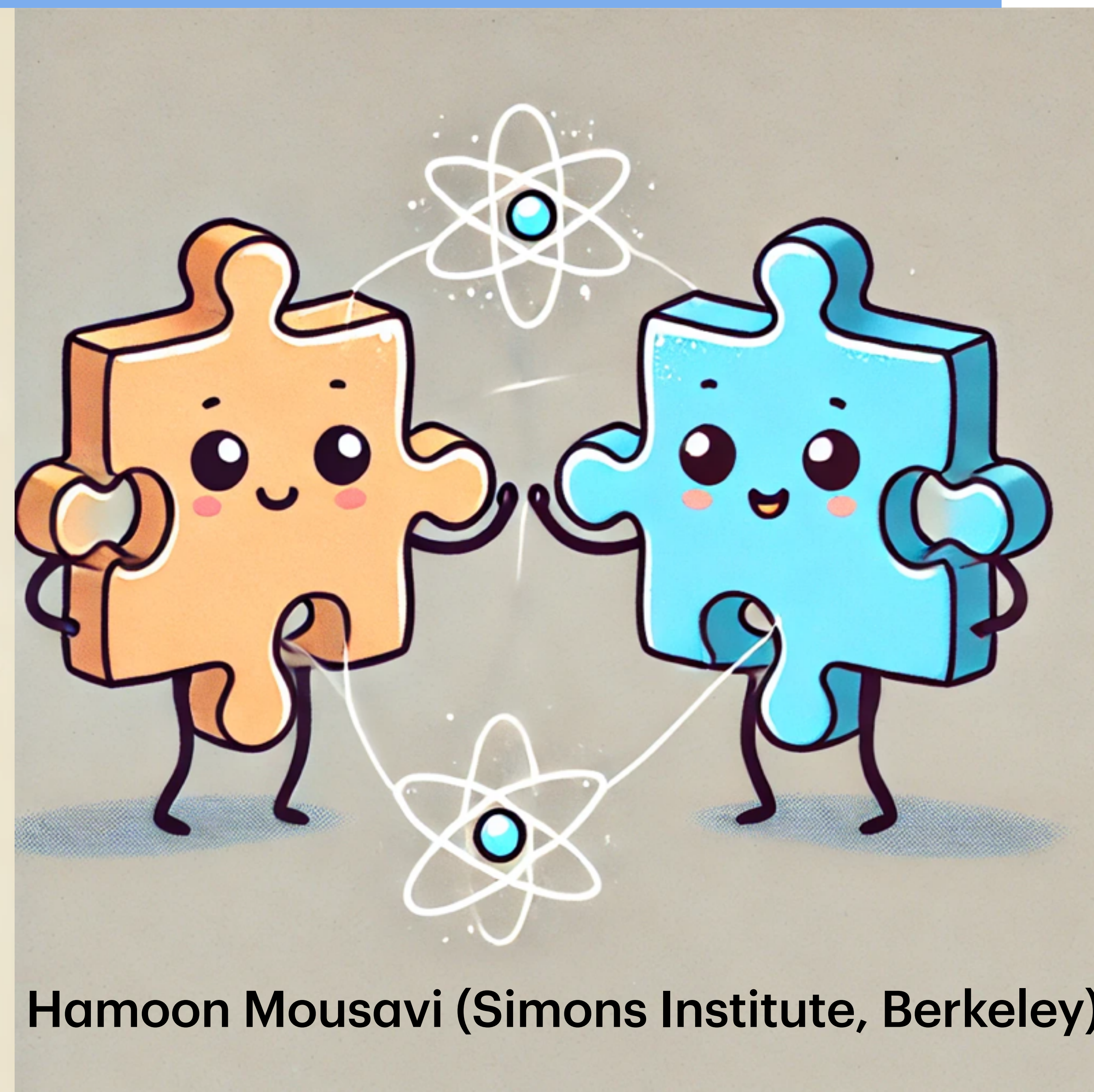
Constraint Satisfaction in the Quantum World

The role of noncommutativity





Hamoon Mousavi (Simons Institute, Berkeley)

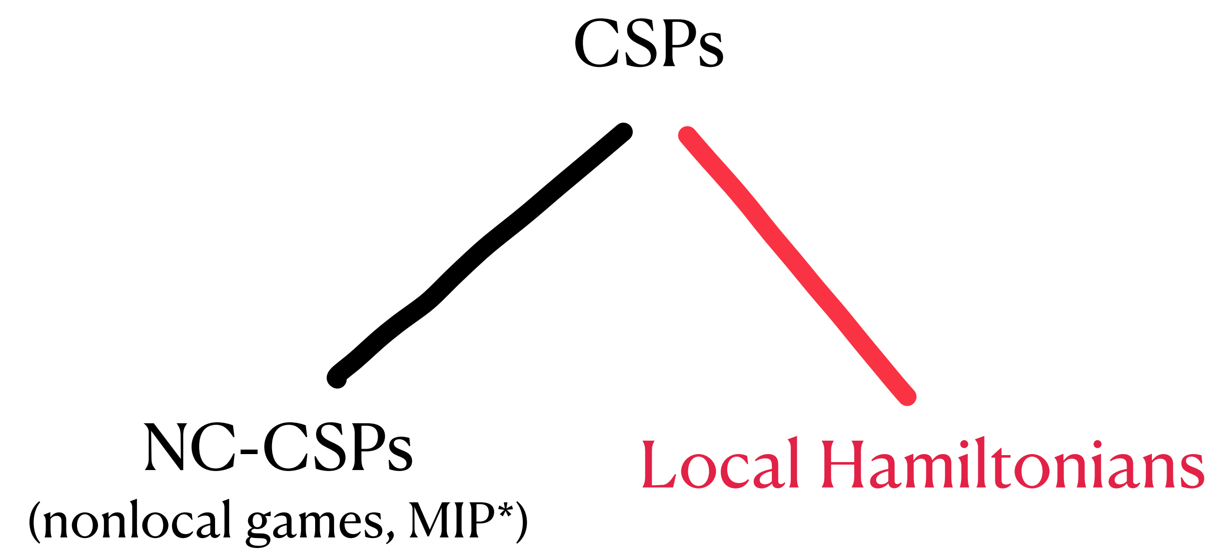
Goal

CSPs

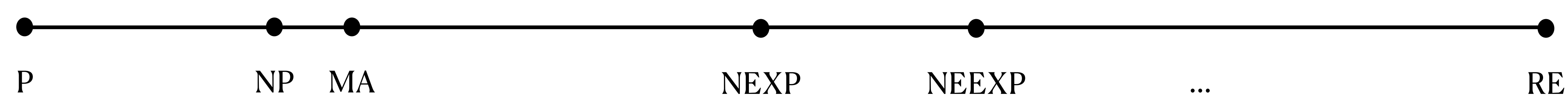
NC-CSPs
(nonlocal games, MIP*)

Local Hamiltonians

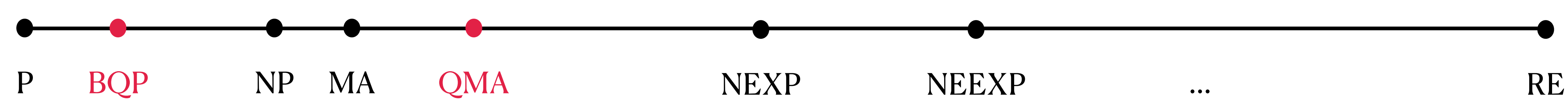
Goal



NC-CSPs



Local Hamiltonian



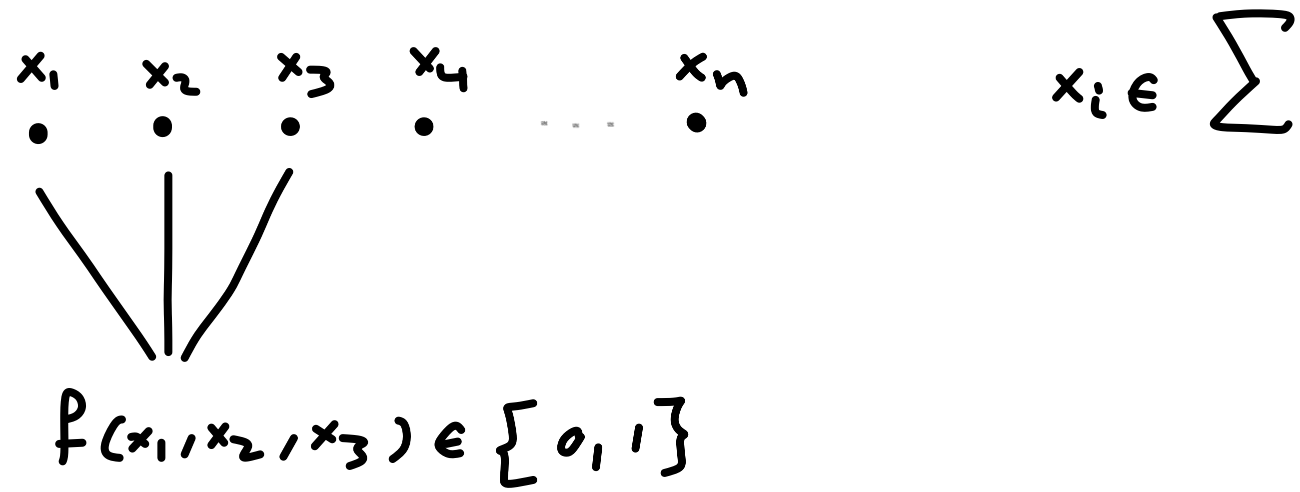
NC-CSPs

CSPs

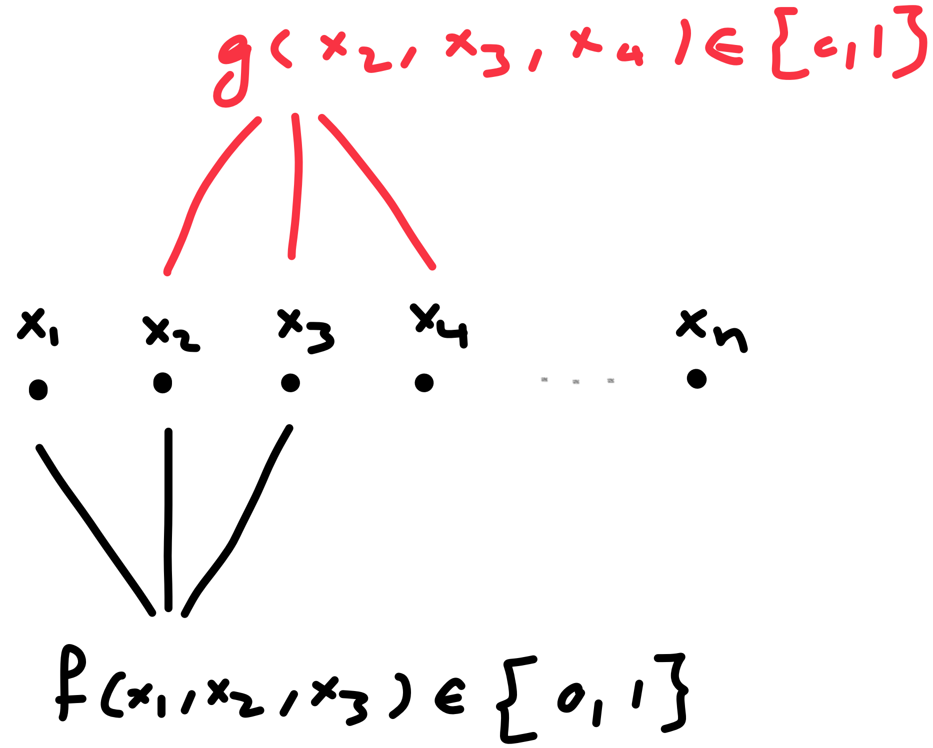
$x_1 \quad x_2 \quad x_3 \quad x_4 \quad \cdots \quad x_n$

$x_i \in \Sigma$

CSPs

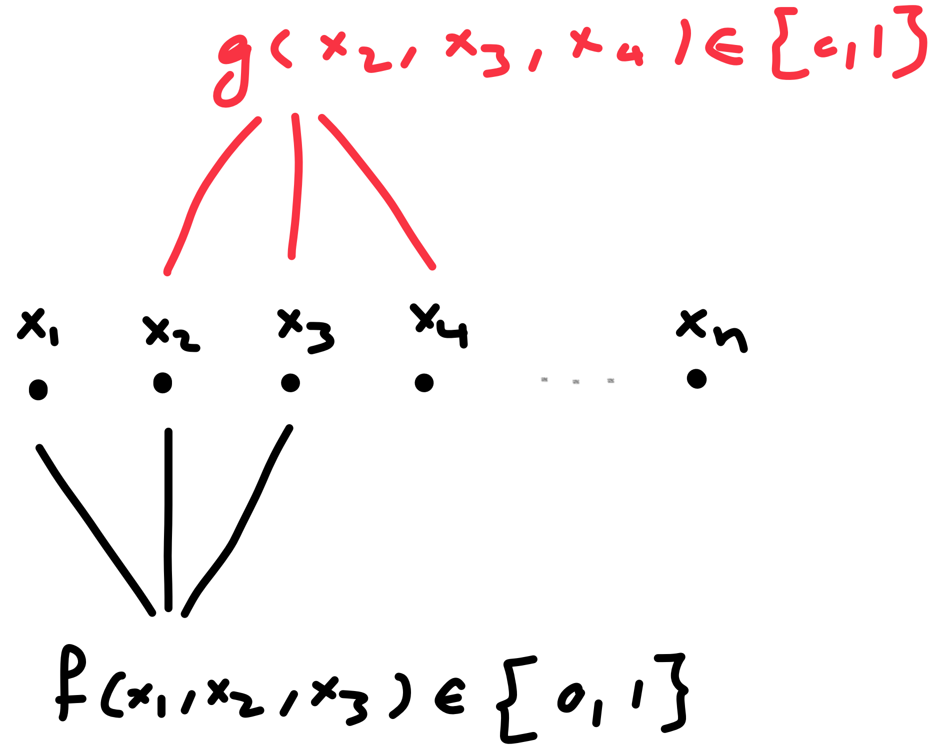


CSPs



$x_i \in \Sigma$

CSPs



$x_i \in \Sigma$

$$\max \quad f(x_1, x_2, x_3) + g(x_2, x_3, x_4) + \cdots$$
$$\text{s.t.} \quad x_i \in \Sigma$$

CSPs

$$\max \quad f(x_1, x_2, x_3) + g(x_2, x_3, x_4) + \cdots$$
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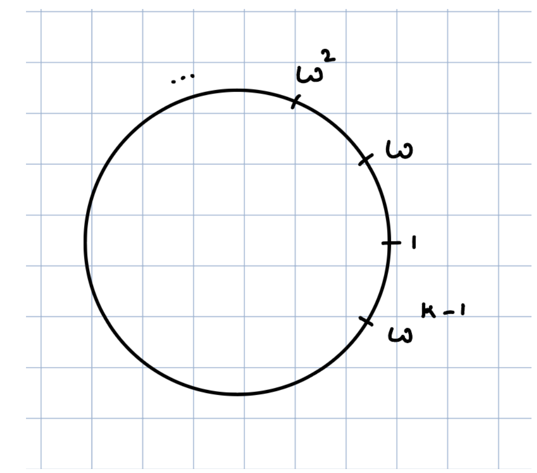
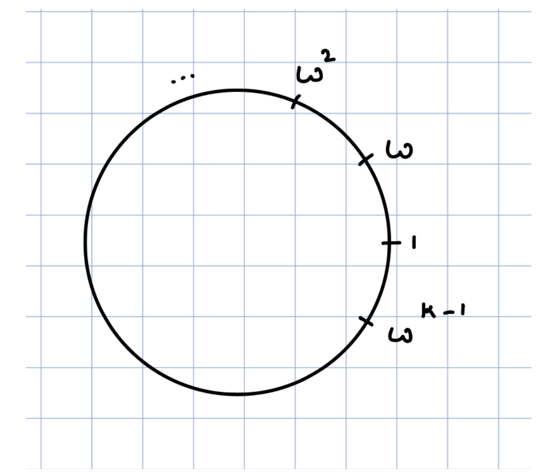
$$\Sigma = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\}$$
$$f : \Sigma^3 \to \{0,1\}$$



CSPs

$$\max \quad f(x_1, x_2, x_3) + g(x_2, x_3, x_4) + \cdots$$
$$\text{s.t.} \quad x_i \in \Sigma$$

$$\Sigma = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\}$$
$$f : \Sigma^3 \to \{0, 1\}$$
$$f(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_2^2 + \cdots$$



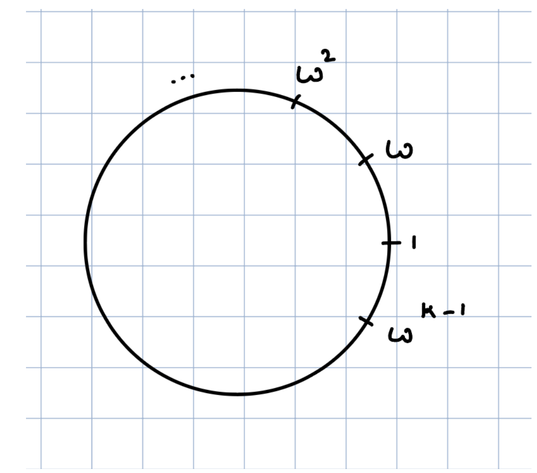
CSPs

$$\begin{aligned} \max \quad & f(x_1, x_2, x_3) + g(x_2, x_3, x_4) + \cdots \\ \text{s.t.} \quad & x_i \in \Sigma \end{aligned}$$

$\Sigma = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\}$

$f : \Sigma^3 \to [0,1]$

$f(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_2^2 + \cdots$

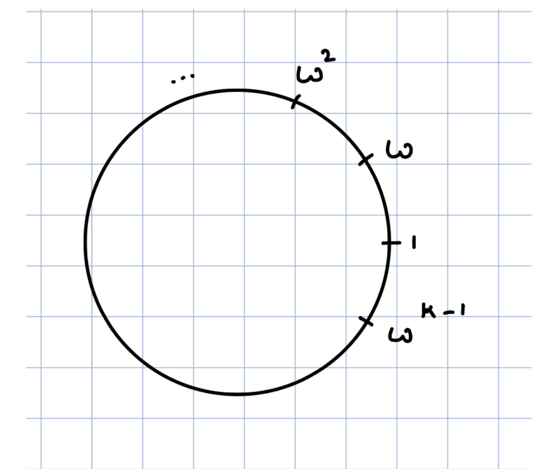


$$\begin{aligned} \max \quad & c_1' x_1 + c_2' x_2 + c_3' x_3 + c_4' x_1 x_2 + c_5' x_1 x_2^2 + \cdots \\ \text{s.t.} \quad & x_i \in \Sigma \end{aligned}$$

CSPs

$$\max \quad f(x_1, x_2, x_3) + g(x_2, x_3, x_4) + \cdots$$
$$\text{s.t.} \quad x_i \in \Sigma$$

$$\Sigma = \{1, \omega, \omega^2, \ldots, \omega^{k-1}\}$$
$$f : \Sigma^3 \to [0,1]$$
$$f(x_1, x_2, x_3) = c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_2^2 + \cdots$$



$$\max \quad c_1' x_1 + c_2' x_2 + c_3' x_3 + c_4' x_1 x_2 + c_5' x_1 x_2^2 + \cdots$$
$$\text{s.t.} \quad x_i \in \Sigma$$

$$\max \quad c_1' x_1 + c_2' x_2 + c_3' x_3 + c_4' x_1 x_2 + c_5' x_1 x_2^2 + \cdots$$
$$\text{s.t.} \quad x_i^k = 1$$

CSPs

$$\max \quad c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_2^2 + \dots$$

$$\text{s.t.} \quad x_i \in \mathbb{C}, \quad |x_i| = 1$$

$$x_i^k = 1$$

CSPs

$$\max \quad c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_2^2 + \cdots$$

$$\text{s.t.} \quad x_i \in \mathbb{C}, \quad |x_i| = 1$$

$$x_i^k = 1$$

NC-CSPs

$$\max \operatorname{tr}\left(c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_1 X_2 + c_5 X_1 X_2^2 + \cdots \right)$$

$$\text{s.t.} \quad d \in \mathbb{N}$$

$$X_i \in \mathcal{U}_d(\mathbb{C})$$

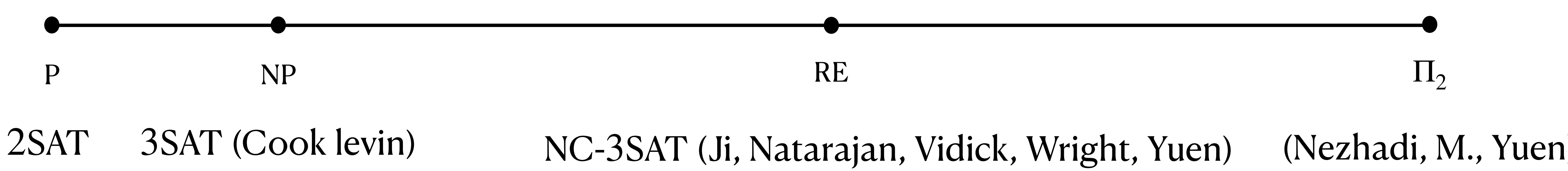
$$X_i^k = 1$$

and commutation relations!

Complexity of NC-CSPs

- Approximating the value of NC-CSPs to within any additive constant is RE-hard
  (Ji, Natarajan, Vidick, Wright, Yuen, 2020)

- Exactly computing the value of NC-CSPs is Π_2-hard
  (Nezhadi, **M.**, Yuen, 2022)



Random Assignments to CSPs

3SAT:

$$(\sim x_3 \lor x_2 \lor x_4) \land (\sim x_3 \lor \sim x_5 \lor x_1) \land (x_3 \lor \sim x_6 \lor \sim x_2)$$

Noncommutative assignments are generalizations of probabilistic assignments

Binary observables are operator generalizations of binary random variables

$$X^*X = I \qquad X^2 = I$$

- ± 1-eigenspaces
- Let x be a binary-outcome random variables: $x \in \{+1, -1\}$

Binary observables are operator generalizations of binary random variables

$x \in \{+1, -1\}$ | X unitary with eigenvalues $+1, -1$

- $\frac{1+x}{2}$ | $\frac{I+X}{2}$
- $\mathbb{E}\frac{1+x}{2}$ | $tr(\frac{I+X}{2})$
- $\mathbb{E}\frac{1+x}{2}\frac{1+y}{2}$ | $tr(\frac{I+X}{2}\frac{1+Y}{2})$

CSPs

$$\max \mathbb{E}\, c_1 x_1 + c_2 x_2 + c_3 x_3 + c_4 x_1 x_2 + c_5 x_1 x_2^2 + \dots$$

s.t. x_i are k-outcome r.v.

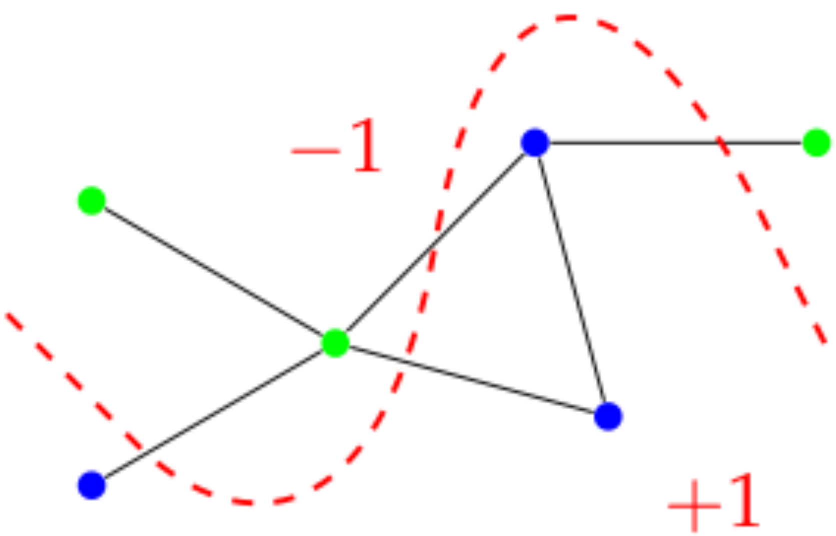
NC-CSPs

$$\max \operatorname{tr}\left(c_1 X_1 + c_2 X_2 + c_3 X_3 + c_4 X_1 X_2 + c_5 X_1 X_2^2 + \dots\right)$$

s.t. X_i are k-outcome observables

Max-Cut

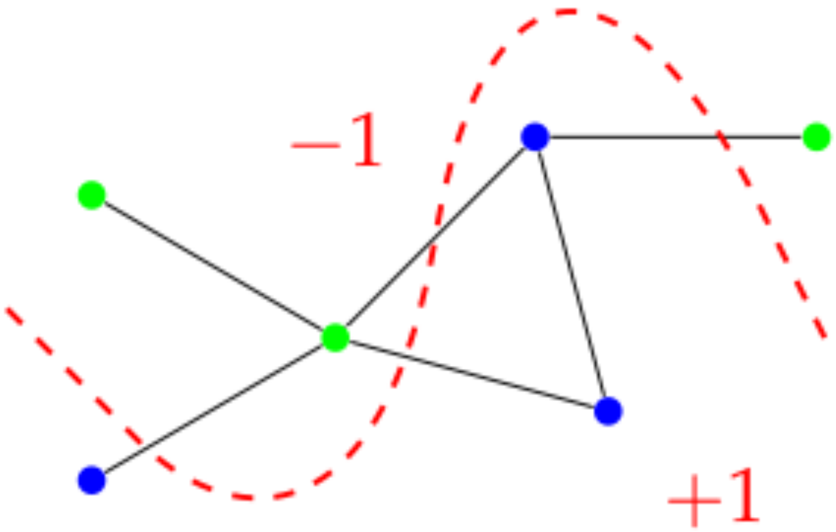
Max-Cut



$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$$

$$\text{subject to:} \quad x_i \in \{-1, +1\}.$$

Max-Cut



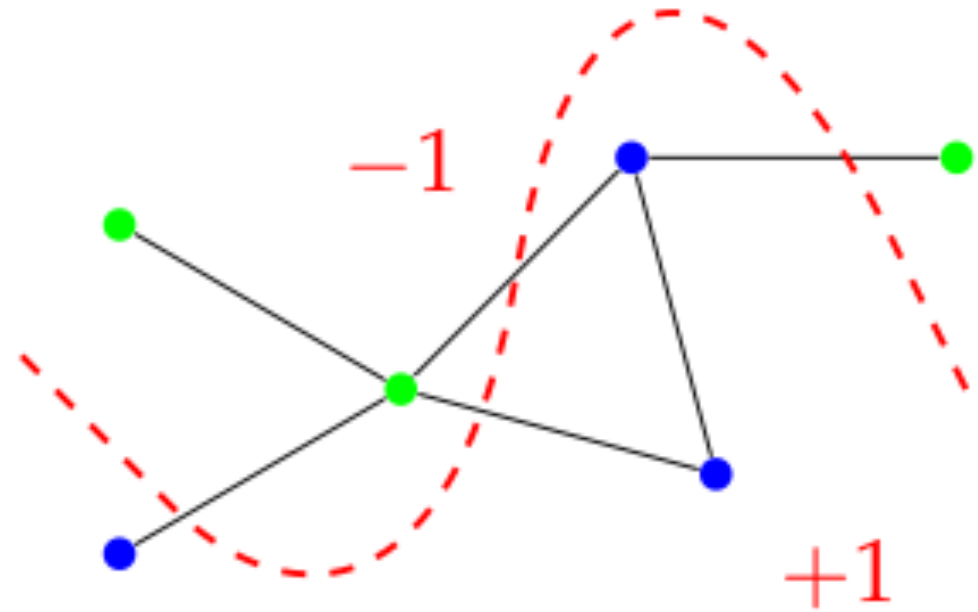
maximize: $\sum_{(i,j)\in E} \frac{1 - x_i x_j}{2}$

subject to: $x_i \in \{-1, +1\}$.

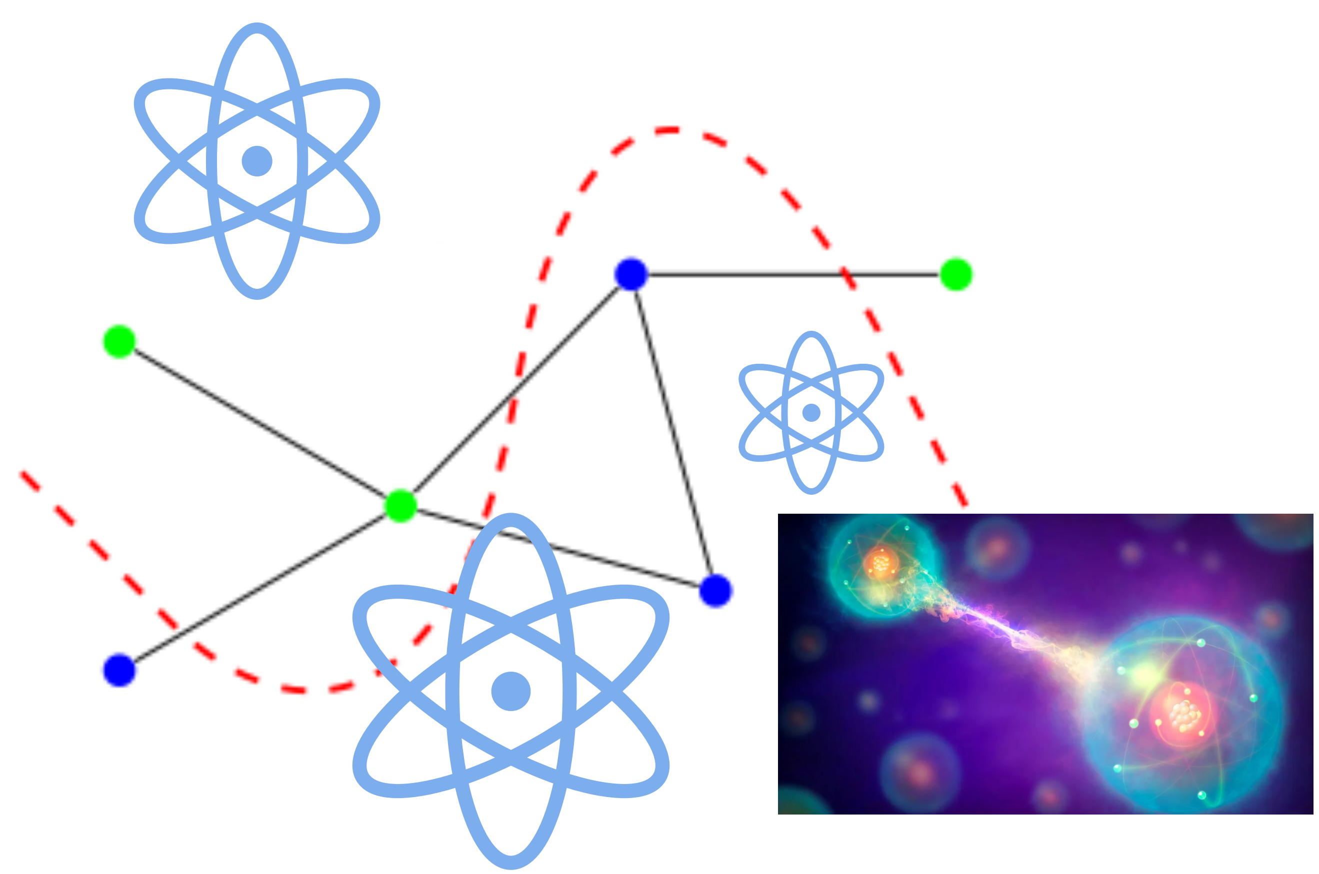
Noncommutative Max-Cut

$$\max \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

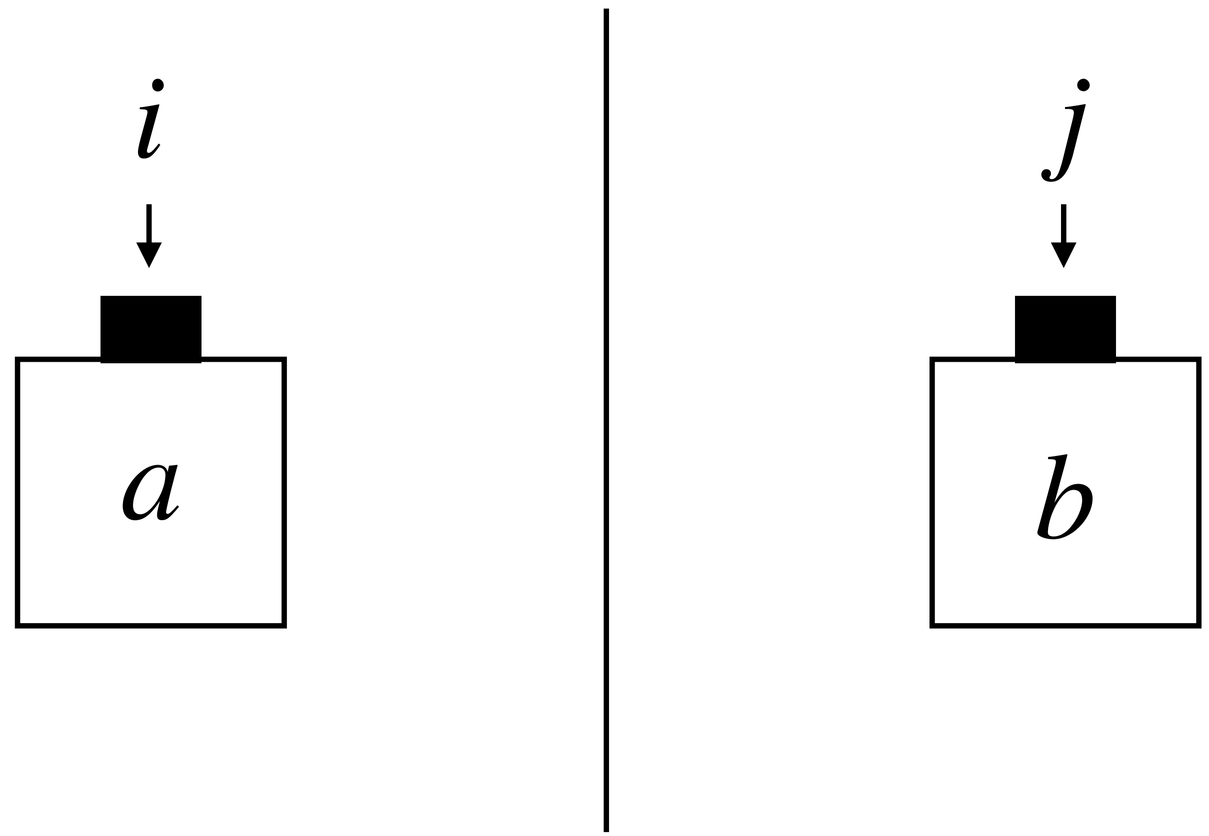


But what does a noncommutative cut look like?



Operational interpretation of NC-CSPs: Multiprover interactive proofs (nonlocal games)

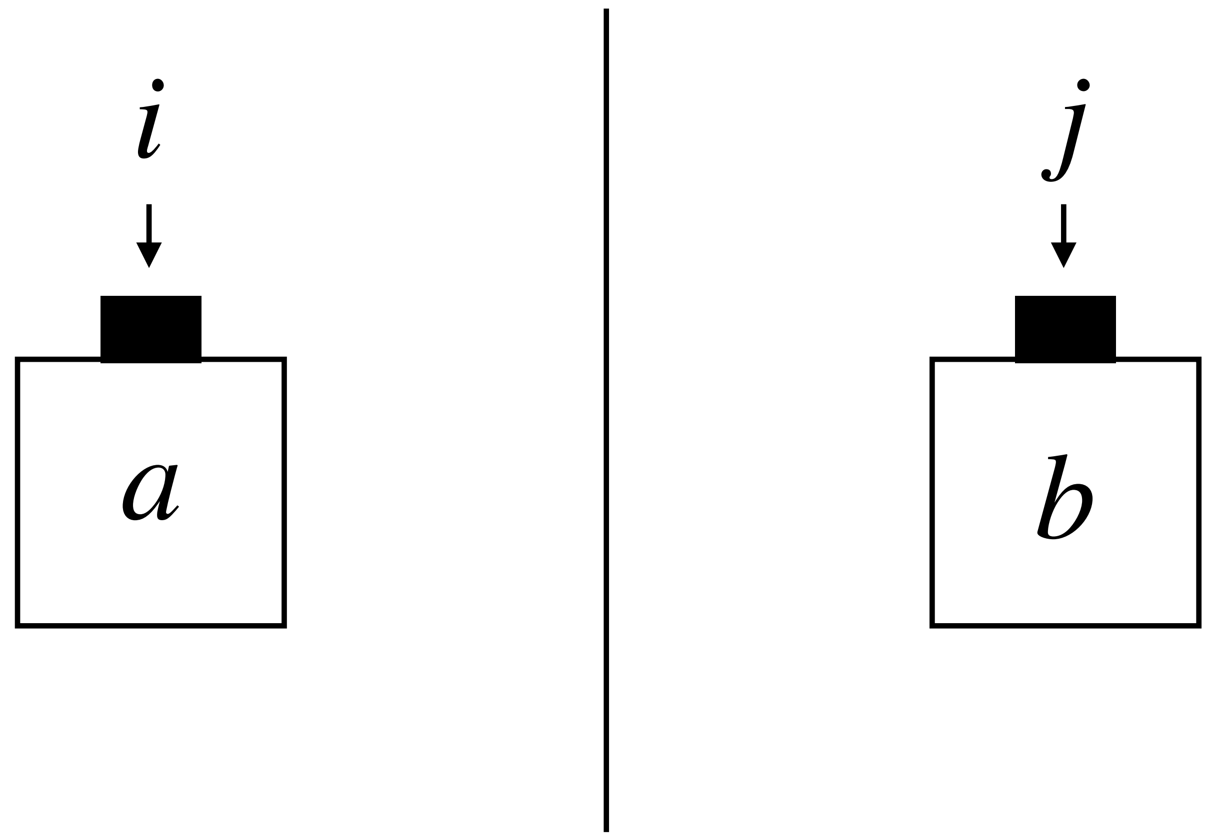
Operational Interpretation of Noncommutative Cuts



$$i, j \in V,$$

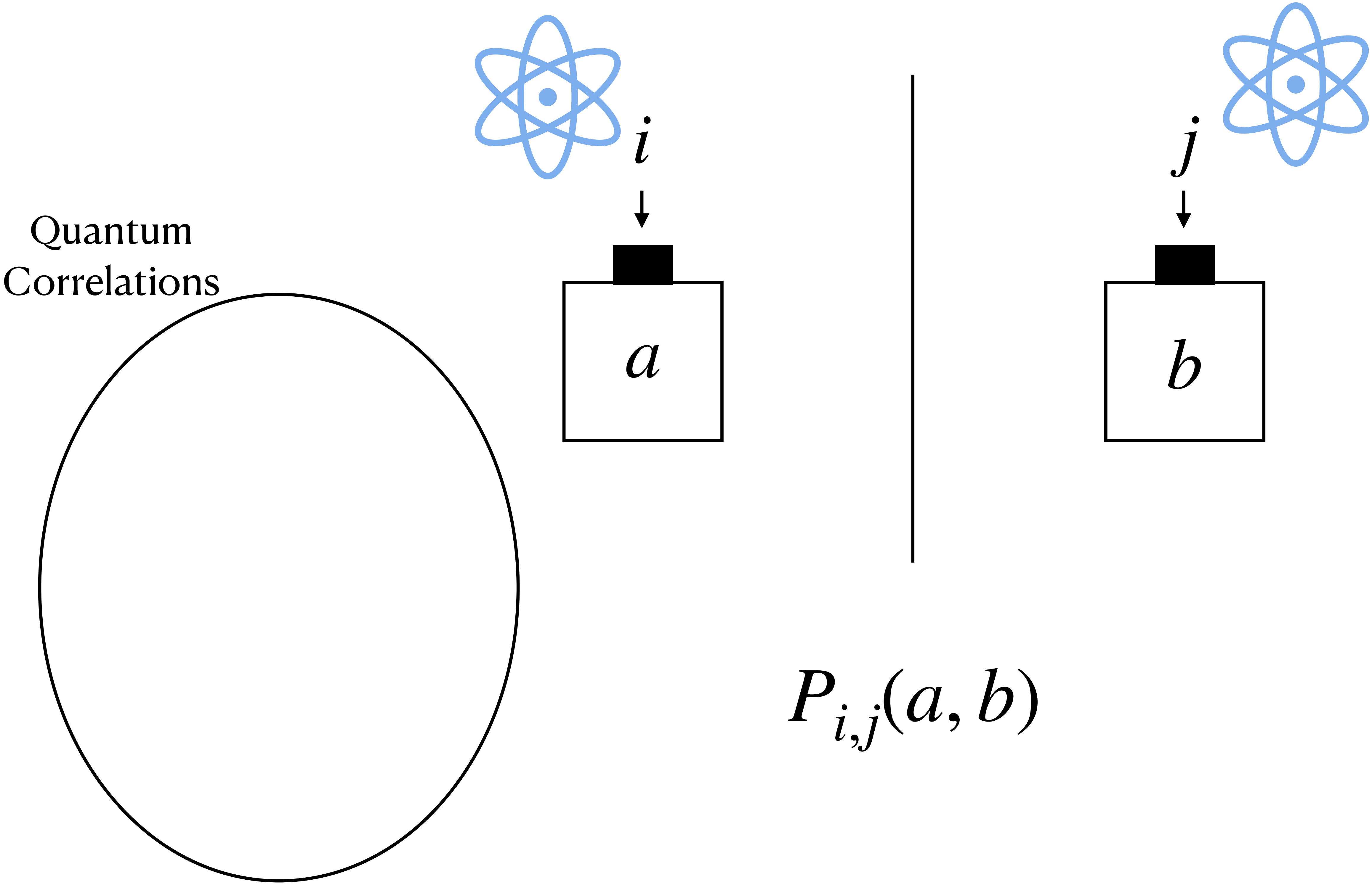
$$a, b \in \{+1, -1\}$$

Correlations

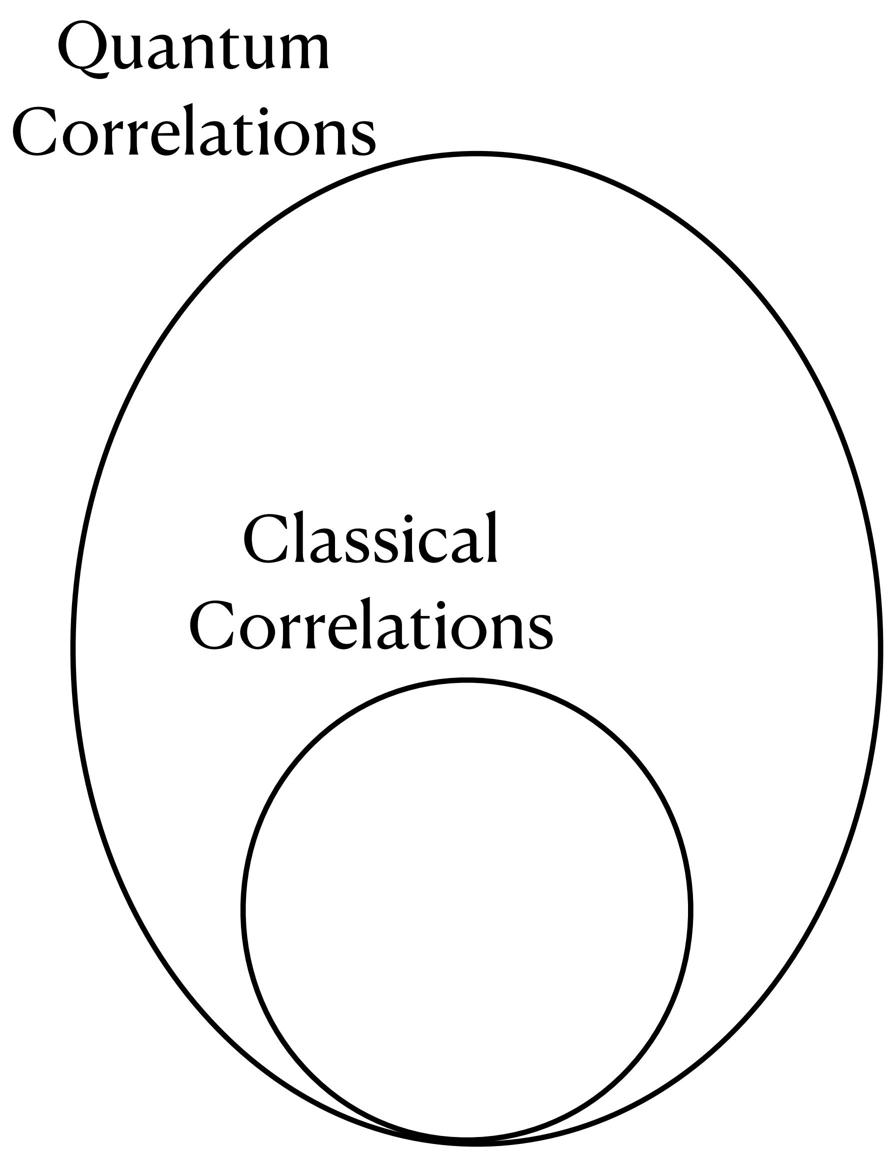


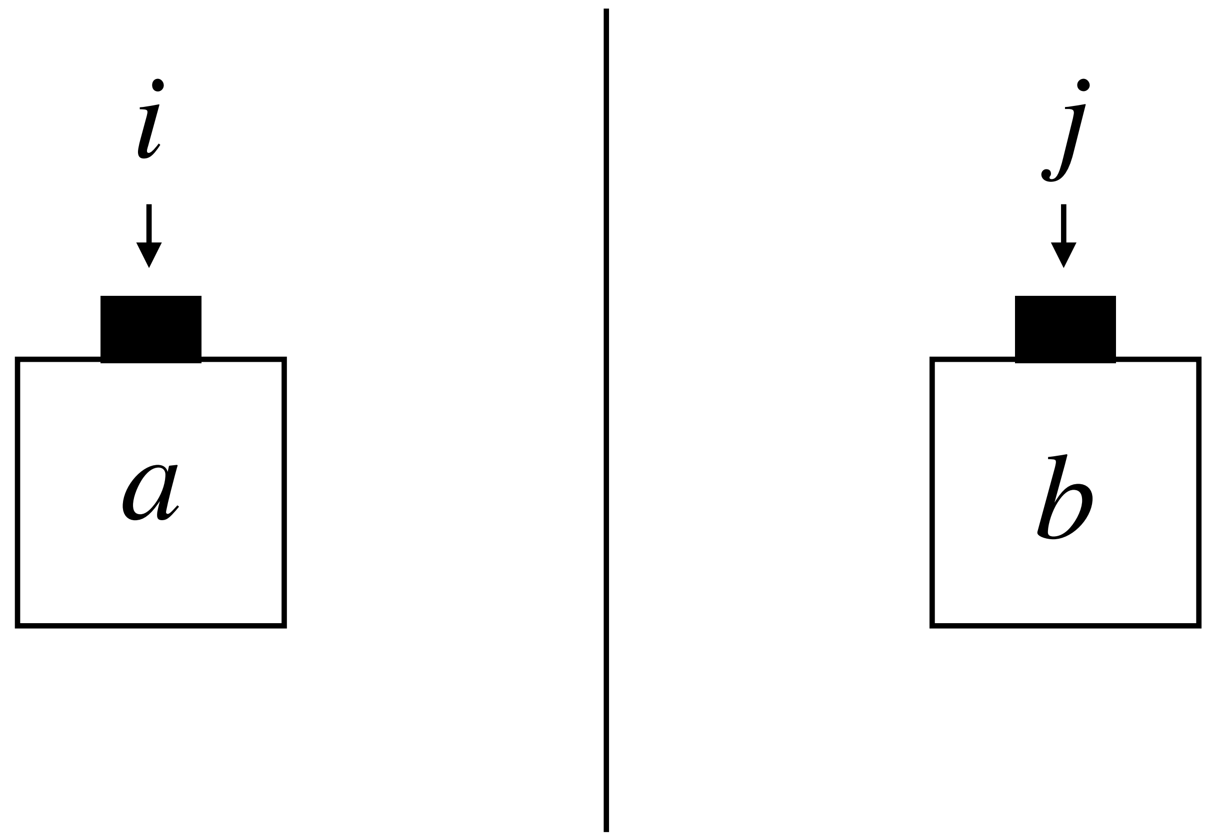
$$P_{i,j}(a,b)$$

Quantum Correlations



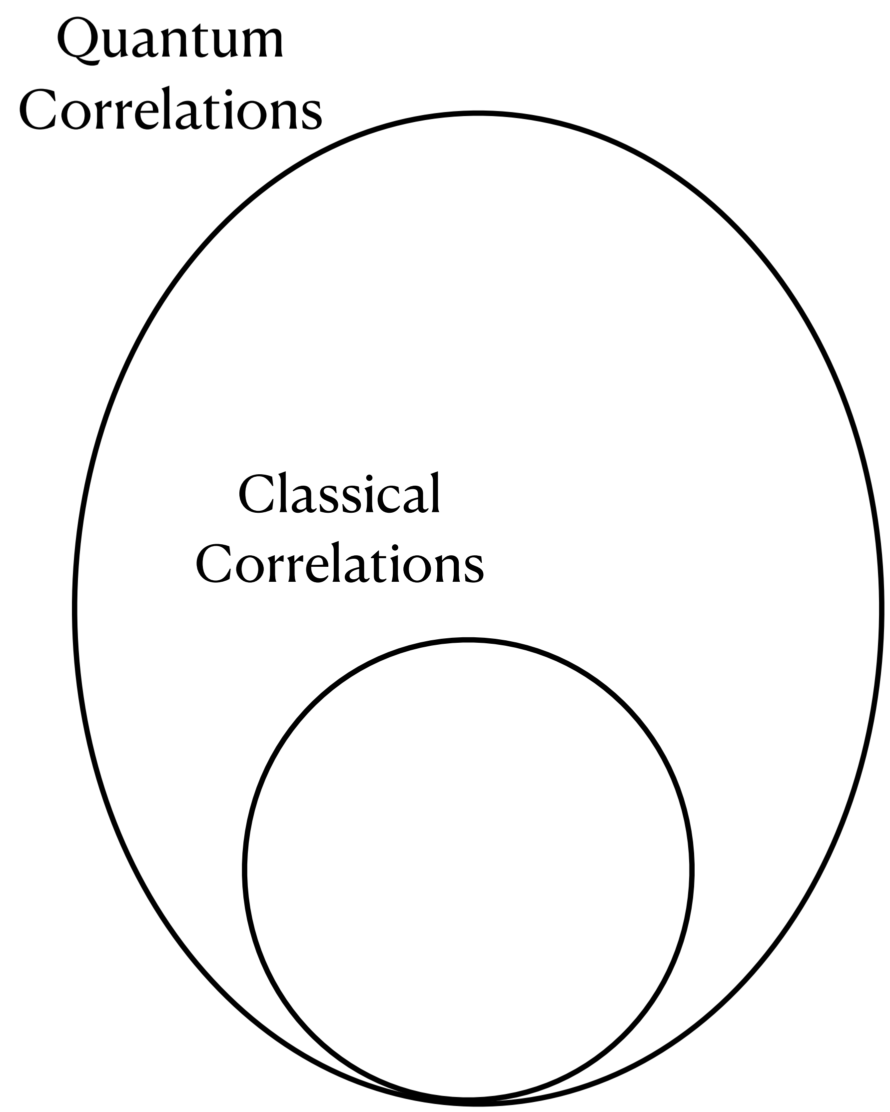
Classical Correlations



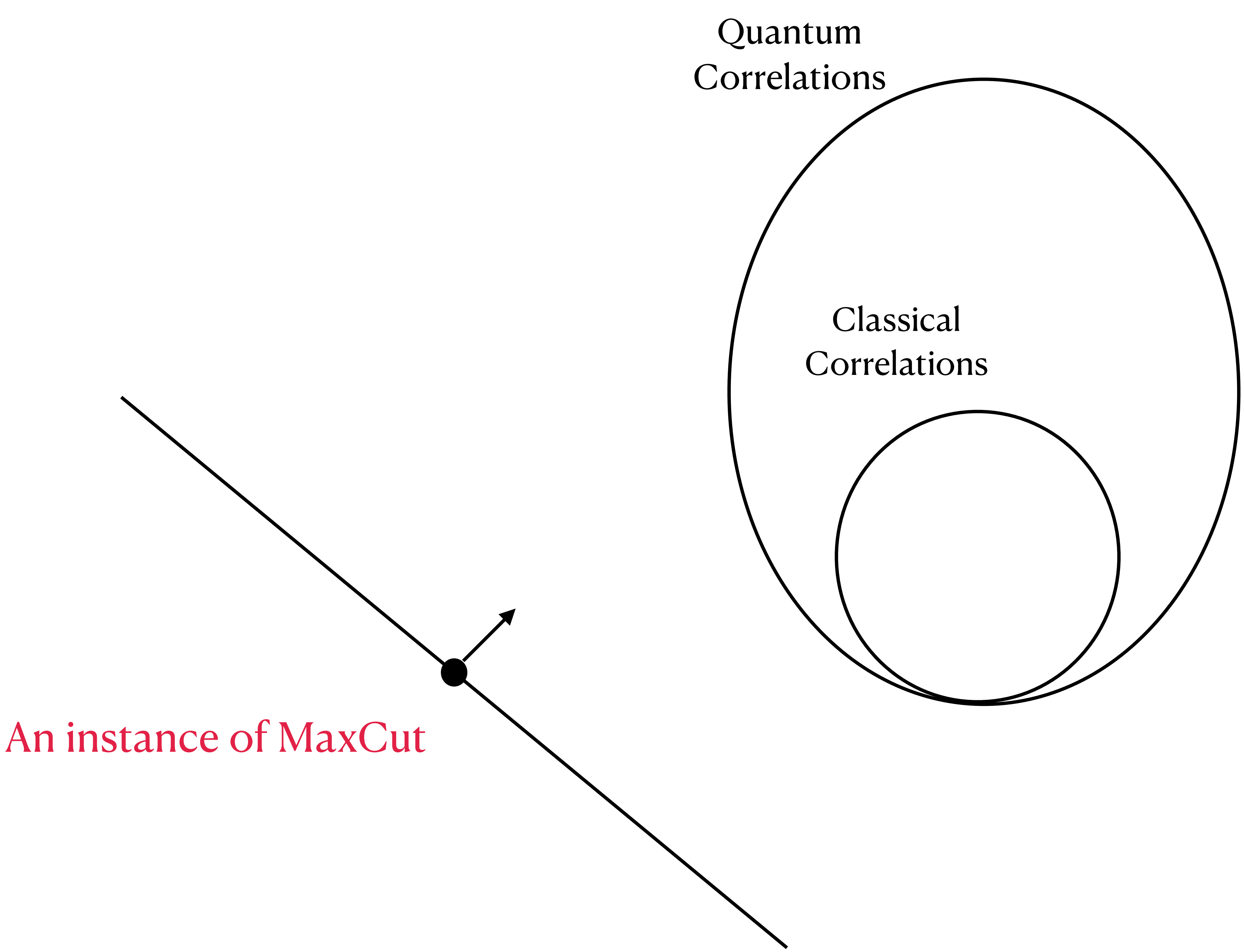


$$P_{i,j}(a, b)$$

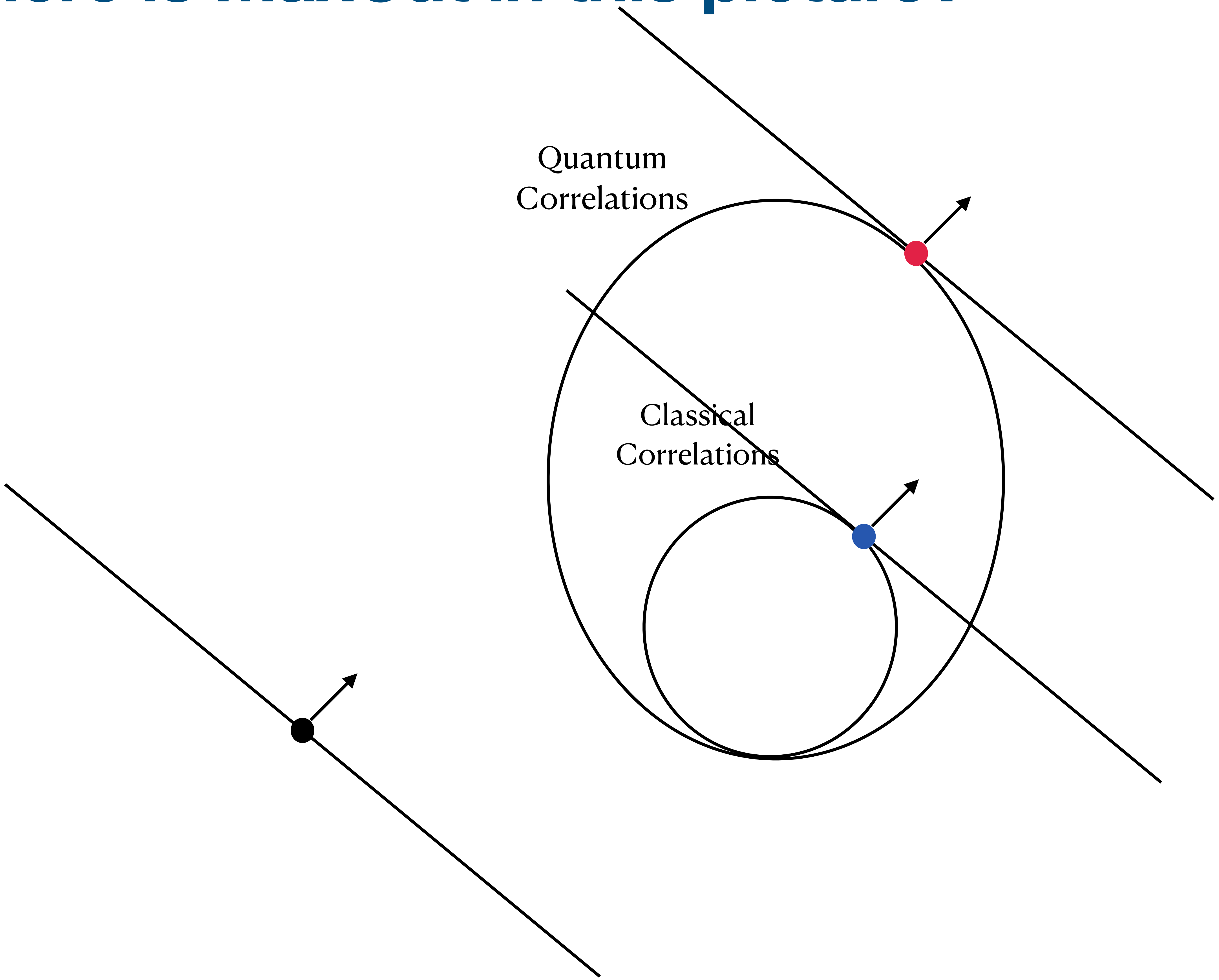
Where is MaxCut in this picture?



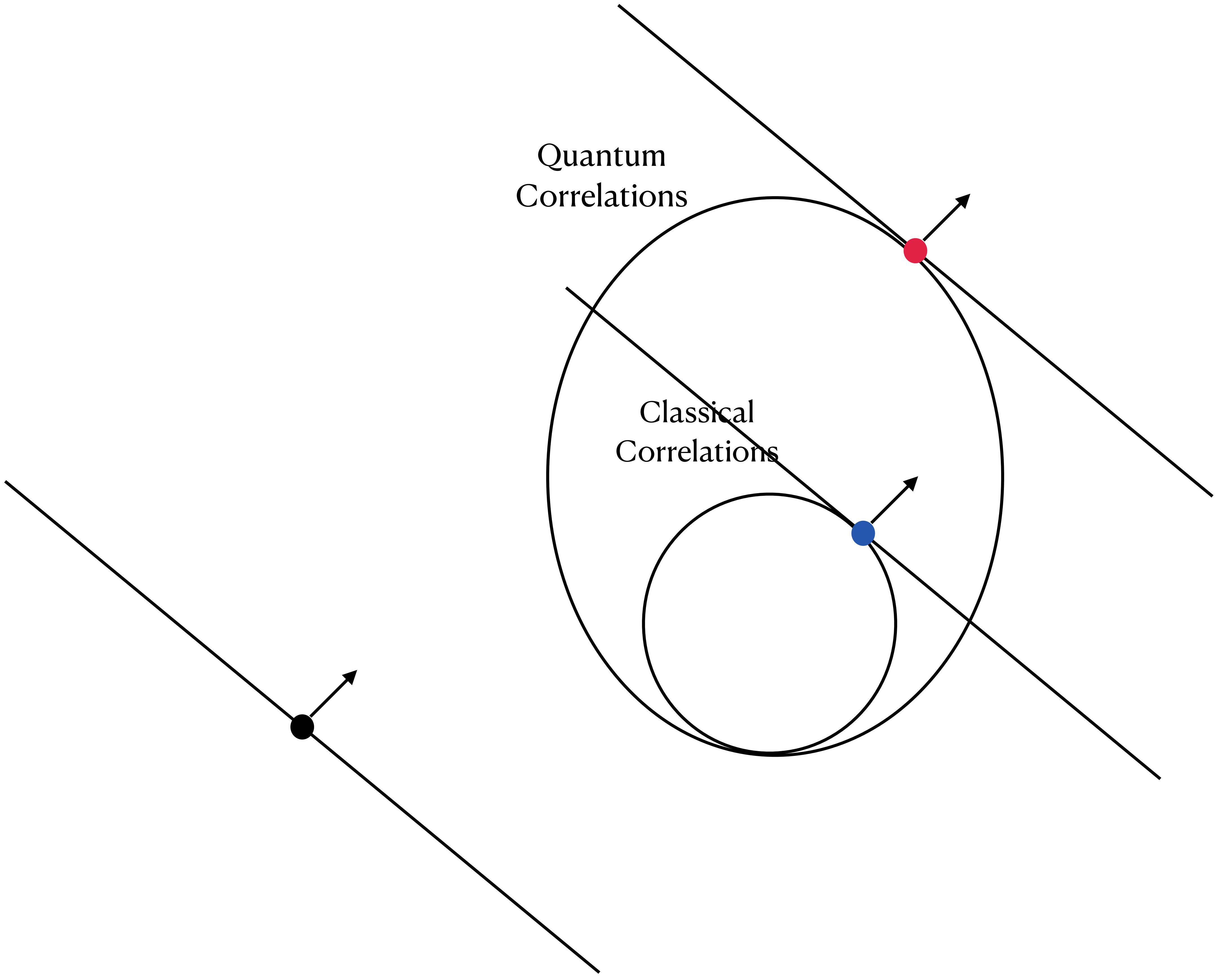
Where is MaxCut in this picture?



Where is MaxCut in this picture?



The 2022 Nobel Prize in Physics awarded to Alain Aspect, John F. Clauser, and Anton Zeilinger



Hardness of Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

- Karp 1972: MaxCut is NP-Complete
-

Hardness of Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. $\quad X_i \quad$ is unitary with ± 1 eigenvalues

- Karp 1972: MaxCut is NP-Complete
- Tsirelson 1980: NC-MaxCut is in P

Hardness of Noncommutative MaxCut

$$\max \quad \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

- Karp 1972: MaxCut is NP-Complete
- Tsirelson 1980: NC-MaxCut is in P
- The best classical algorithm is SDP rounding by Goemans and Williamson
- Tsirelson's algorithm is an operator generalization

Sample vector $\vec{r}$ from the unit sphere

Let x_i be the sign of $\langle \vec{r}, \vec{x}_i \rangle$

$$\max \sum \frac{w_{ij}}{2}(1 - x_i x_j) \quad \leq \quad \max \sum \frac{w_{ij}}{2}(1 - \langle X_i, X_j \rangle) \quad \leq \quad \max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

$$\text{s.t} \quad x_i^2 = 1 \qquad \text{s.t} \quad X_i^2 = X_i^* X_i = 1 \qquad \text{s.t} \quad \langle \vec{x}_i, \vec{x}_i \rangle = 1$$

Sample vector $\vec{r}$ from the unit sphere

Let x_i be the sign of $\langle \vec{r}, \vec{x}_i \rangle$

$$\max \sum \frac{w_{ij}}{2}(1 - x_i x_j) \quad \leq \quad \max \sum \frac{w_{ij}}{2}(1 - \langle X_i, X_j \rangle) \quad \leq \quad \max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

s.t $x_i^2 = 1$ s.t $X_i^2 = X_i^* X_i = 1$ s.t $\langle \vec{x}_i, \vec{x}_i \rangle = 1$

$$\overrightarrow{x_i} = (\alpha_1, \ldots, \alpha_n) \longrightarrow X = \alpha_1 \sigma_1 + \cdots + \alpha_n \sigma_n$$

Sample vector $\vec{r}$ from the unit sphere

Let x_i be the sign of $\langle \vec{r}, \vec{x}_i \rangle$

$$\max \sum \frac{w_{ij}}{2}(1 - x_i x_j) \quad \leq \quad \max \sum \frac{w_{ij}}{2}(1 - \langle X_i, X_j \rangle) \quad = \quad \max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

s.t $x_i^2 = 1$ s.t $X_i^2 = X_i^* X_i = 1$ s.t $\langle \vec{x}_i, \vec{x}_i \rangle = 1$

$$\overrightarrow{x_i} = (\alpha_1, \ldots, \alpha_n) \longrightarrow X = \alpha_1 \sigma_1 + \cdots + \alpha_n \sigma_n$$

Goemans-Williamson

Max-Cut

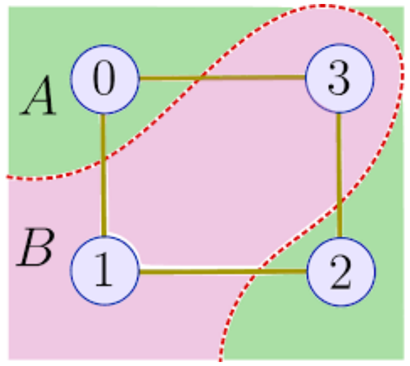
$$\max \sum \frac{w_{ij}}{2}(1 - x_i x_j)$$

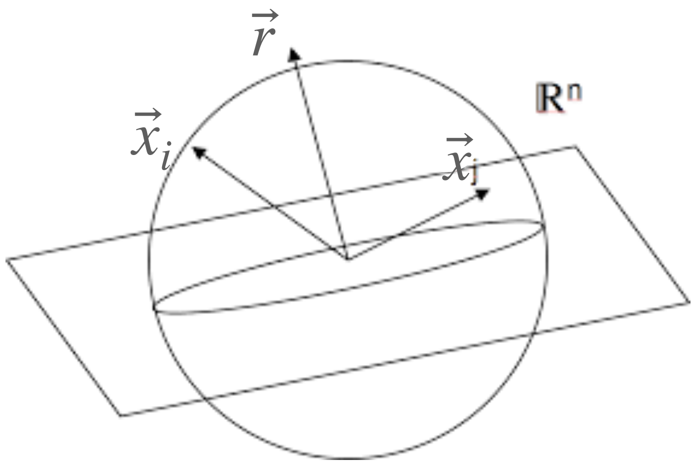
s.t. $x_i^2 = 1$

Max-Cut-SDP

$$\max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

s.t. $\langle \vec{x}_i, \vec{x}_i \rangle = 1$





Goemans-Williamson Theorem

$0.878 \times$ **Max-Cut-SDP** $\leq$ **Max-Cut** $\leq$ **Max-Cut-SDP**

Hyperplane rounding scheme

Sample vector $\vec{r}$ from the unit sphere

Let x_i be the sign of $\langle \vec{r}, \vec{x}_i \rangle$

$$\mathbb{E}\left(\frac{1 - x_i x_j}{2}\right) \geq 0.878 \frac{1 - \langle \vec{x}_i, \vec{x}_j \rangle}{2}$$

Tsirelson's theorem (operator extension of Goemans-Williamson)

NC-Max-Cut

$$\max Tr \sum \frac{w_{ij}}{2}(1 - X_i X_j)$$

s.t. $X_i^2 = X_i^* X_i = 1$

Tsirelson's theorem

NC-Max-Cut = Max-Cut-SDP

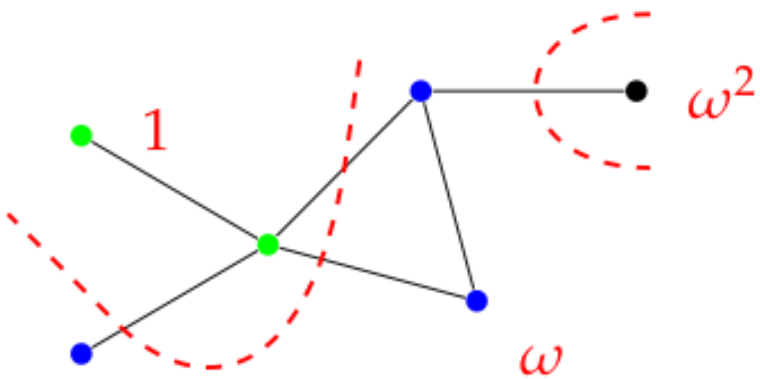
Max-Cut-SDP

$$\max \sum \frac{w_{ij}}{2}(1 - \langle \vec{x}_i, \vec{x}_j \rangle)$$

s.t. $\langle \vec{x}_i, \vec{x}_i \rangle = 1$

$$\vec{x} = (x_1, \ldots, x_n) \longrightarrow X = x_1\sigma_1 + \cdots + x_n\sigma_n$$

Max-3-Cut



(a) Example of a partition of vertices into three subsets

$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{2 - x_i^* x_j - x_j^* x_i}{3}$$

$$\text{subject to:} \quad x_i \in \{1, \omega, \omega^2\},$$

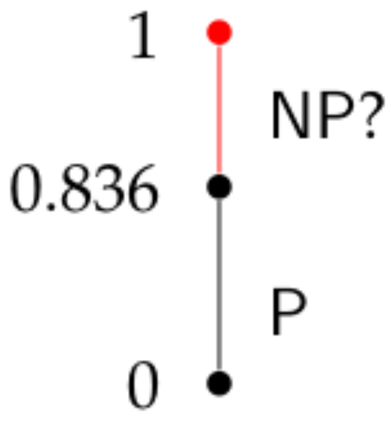
(b) Max-3-Cut as a polynomial optimization

Noncommutative Max-3-Cut

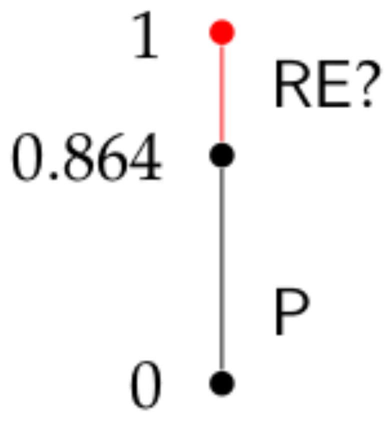
$$\text{maximize:} \quad \sum_{(i,j)\in E} \frac{2 - \langle X_i, X_j \rangle - \langle X_j, X_i \rangle}{3}$$

$$\text{subject to:} \quad X_i \text{ unitary with eigenvalues } 1, \omega, \omega^2.$$

What about other NC-CSPs?



(a) Max-3-Cut



(b) Noncommutative Max-3-Cut

Frieze and Jerrum

Culf, **M.**, Spirig

But why in CS?

Magic Square

$x_{ij} \in \{+1, -1\}$

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

Perfect Operator Solution

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

$x_{ij} \in \{+1, -1\}$

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$x_{ij} \in \{+1, -1\}$

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Binary alphabet $\{+1, -1\}$ in the classical case $\longrightarrow$ Binary observables

x_{11}	x_{12}	x_{13}	$+1$
x_{21}	x_{22}	x_{23}	$+1$
x_{31}	x_{32}	x_{33}	$+1$
$+1$	$+1$	-1	

$x_{ij} \in \{+1, -1\}$

$\longrightarrow$

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Binary alphabet $\{+1, -1\}$ in the classical case $\longrightarrow$ Binary observables

Binary observables: Unitary operators with $\{+1, -1\}$ eigenvalues

$$O^*O = O^2 = I$$

An operator CSP

$X_{ij}^* X_{ij} = I$

$X_{ij}^2 = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

An operator CSP

$X_{ij}^* X_{ij} = I$

$X_{ij}^2 = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

When restricting to one dimension we recover the classical CSP

Because ± 1 are the only binary observables is one dimension

Perfect Operator Solution: algebraic structure

Mermin 1990 and Peres 1990

$I \otimes X$	$X \otimes I$	$X \otimes X$	$+I$
$Z \otimes I$	$I \otimes Z$	$Z \otimes Z$	$+I$
$Z \otimes X$	$X \otimes Z$	$Y \otimes Y$	$+I$
$+I$	$+I$	$-I$	

Uniqueness of the perfect solution

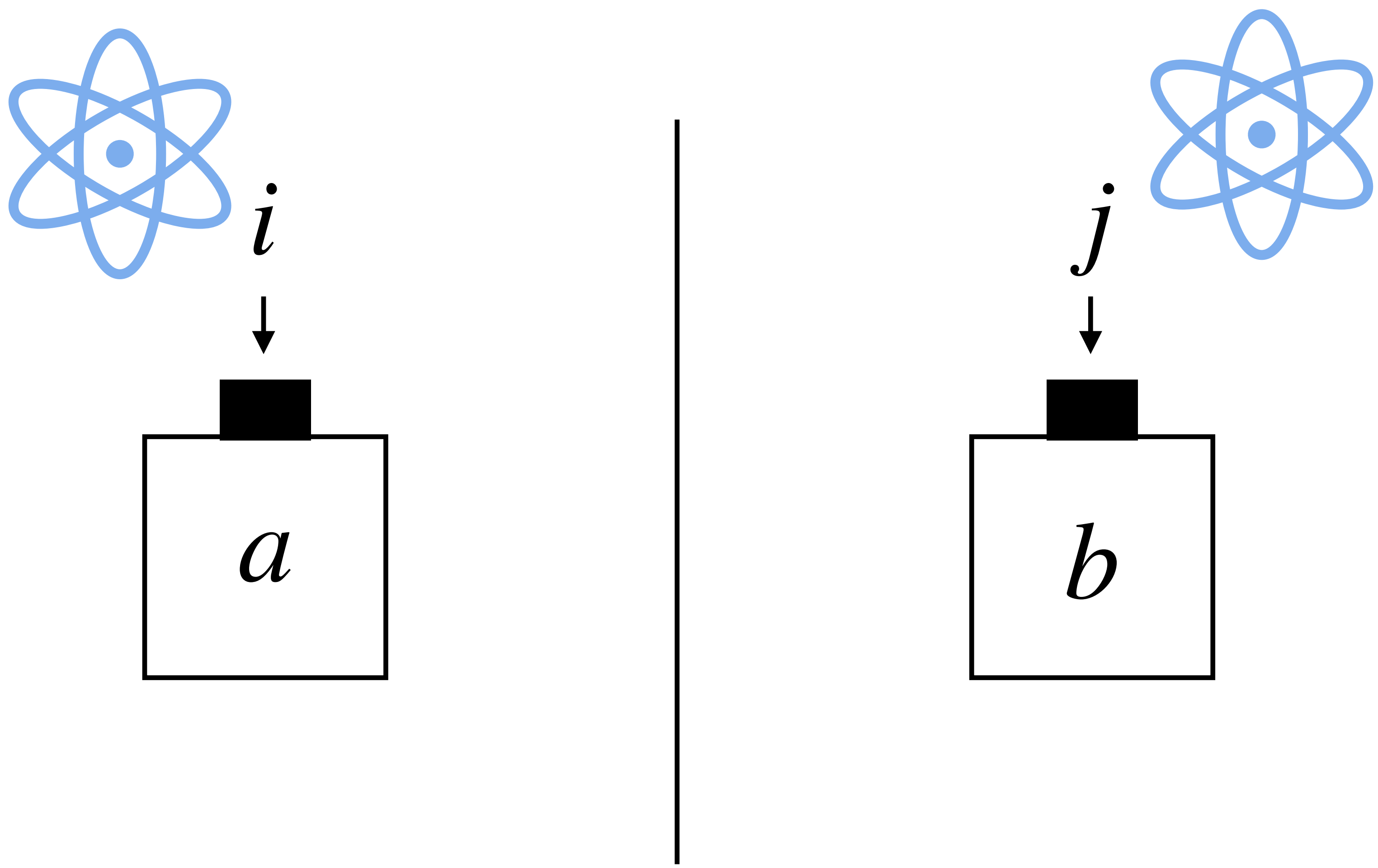
$X_{ij}^* X_{ij} = I$

$X_{ij}^2 = I$

X_{11}	X_{12}	X_{13}	$+I$
X_{21}	X_{22}	X_{23}	$+I$
X_{31}	X_{32}	X_{33}	$+I$
$+I$	$+I$	$-I$	

$X_{11}X_{12} = X_{12}X_{11}, \quad X_{12}X_{21} = -X_{21}X_{12}, \quad \cdots$

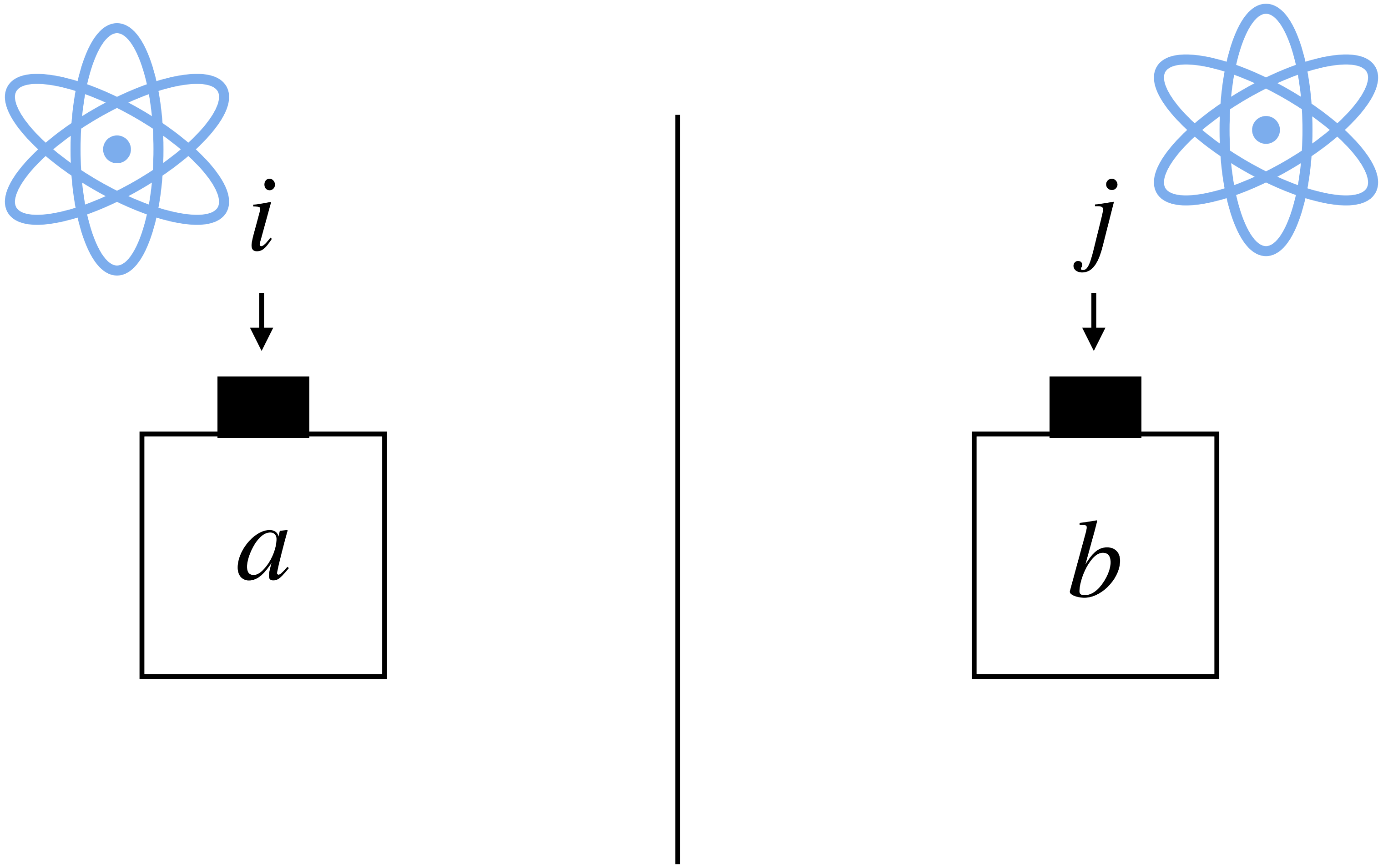
Magic Square Game



$$P_{i,j}(a, b)$$

Magic Square Game

$$XY = -YX$$



$$P_{i,j}(a, b)$$

Hardness of Approximation for NC-CSPs

Hardness front

- PCP theorem: Approximating Label-Cover is NP-hard
(Arora, Safra, Lund, Motwani, Sudan, Szegedy, Raz, Håstad)

- NC-PCP theorem (MIP*=RE): Approximating NC-Label-Cover is RE-hard
(Ji, Natarajan, Vidick, Wright, Yuen 2020)

Hardness front

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- Compare this with the situation for the Local Hamiltonian problem (LH):

  Quantum PCP conjecture: Approximating Local Hamiltonian is QMA-hard

Hardness front

- Similarly UGC has an NC-UGC analogue

- Assuming UGC, approximating MaxCut to better than 0.878 is NP-hard (Khot, Kindler, Mossel, O'Donnell)

- Assuming Q-UGC, approximating Q-MaxCut to better than 0.878 is RE-hard (M., Spirig)

A classical theorem involving NP and CSP

becomes

A theorem that involves RE and NC-CSP

The algebraic nature of CS tools (sum-check protocol, low-degree testing, Fourier analysis on the hypercube)

fits

the algebraic nature of CSPs and NC-CSPs

CSPs: commutative algebras

NC-CSPs: matrix algebras

Local Hamiltonians: not algebraic

NC-CSPs are expressive

P — NP — RE — Π_2

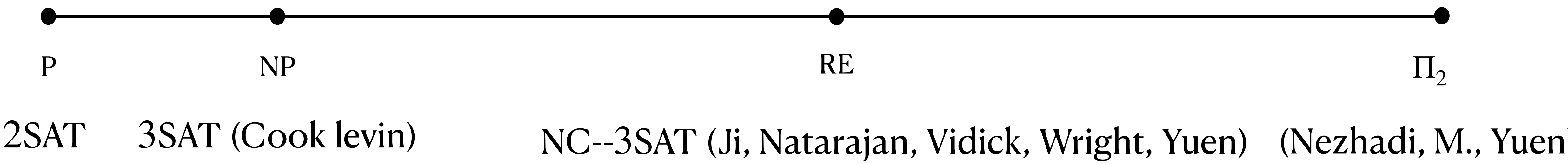
2SAT

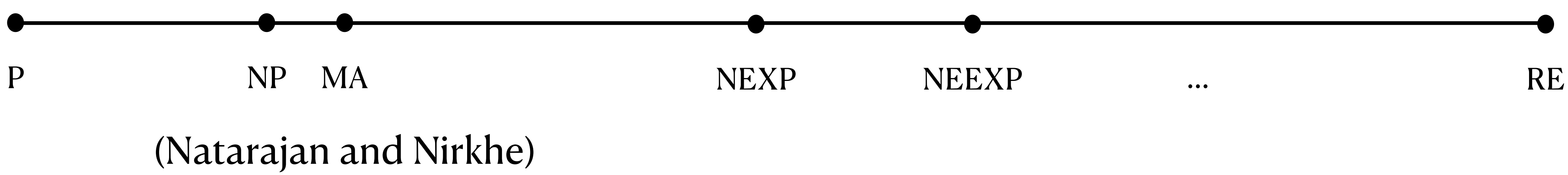
3SAT (Cook levin)

NC--3SAT (Ji, Natarajan, Vidick, Wright, Yuen)

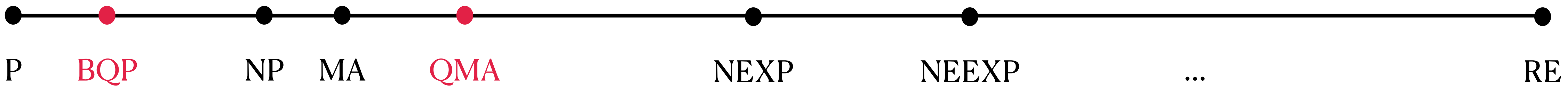
(Nezhadi, M., Yuen)

NC-CSPs are expressive

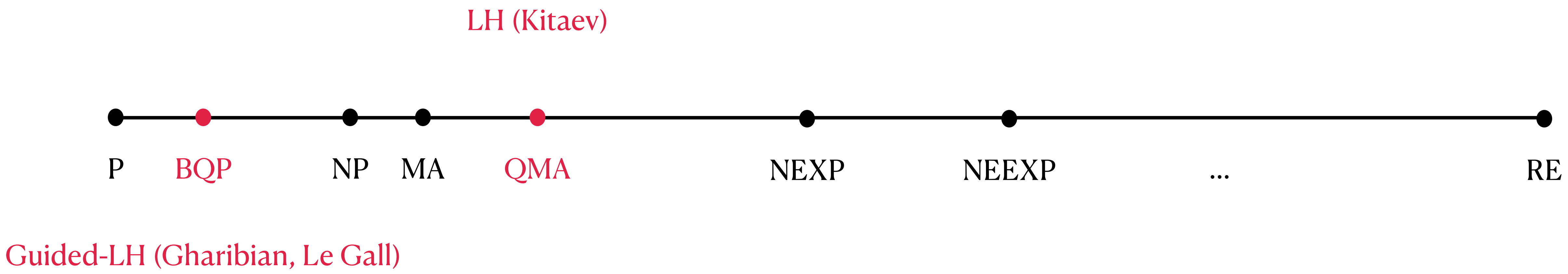




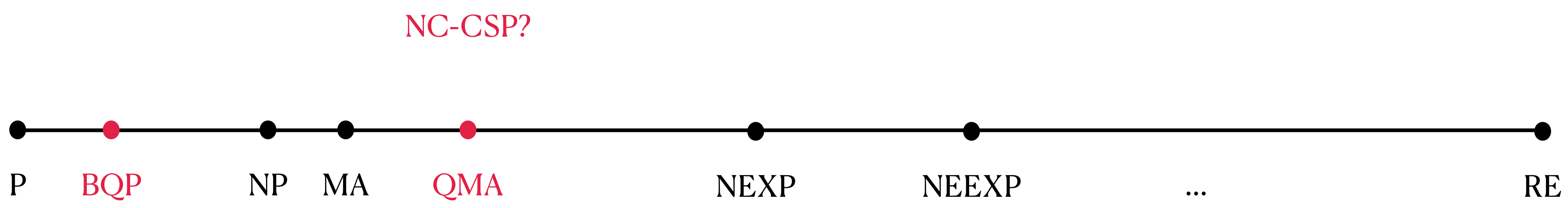
But they skip on quantum complexity classes



Local-Hamiltonian fills the gap

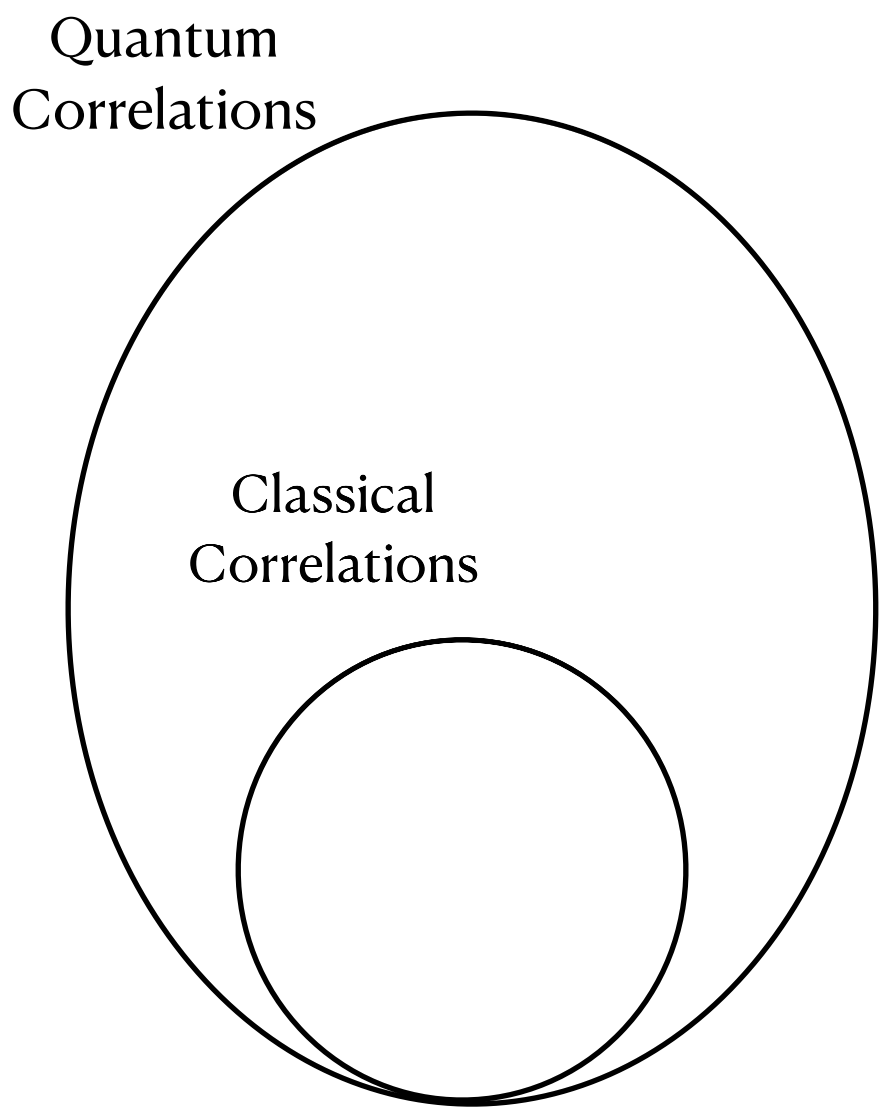


Open problem

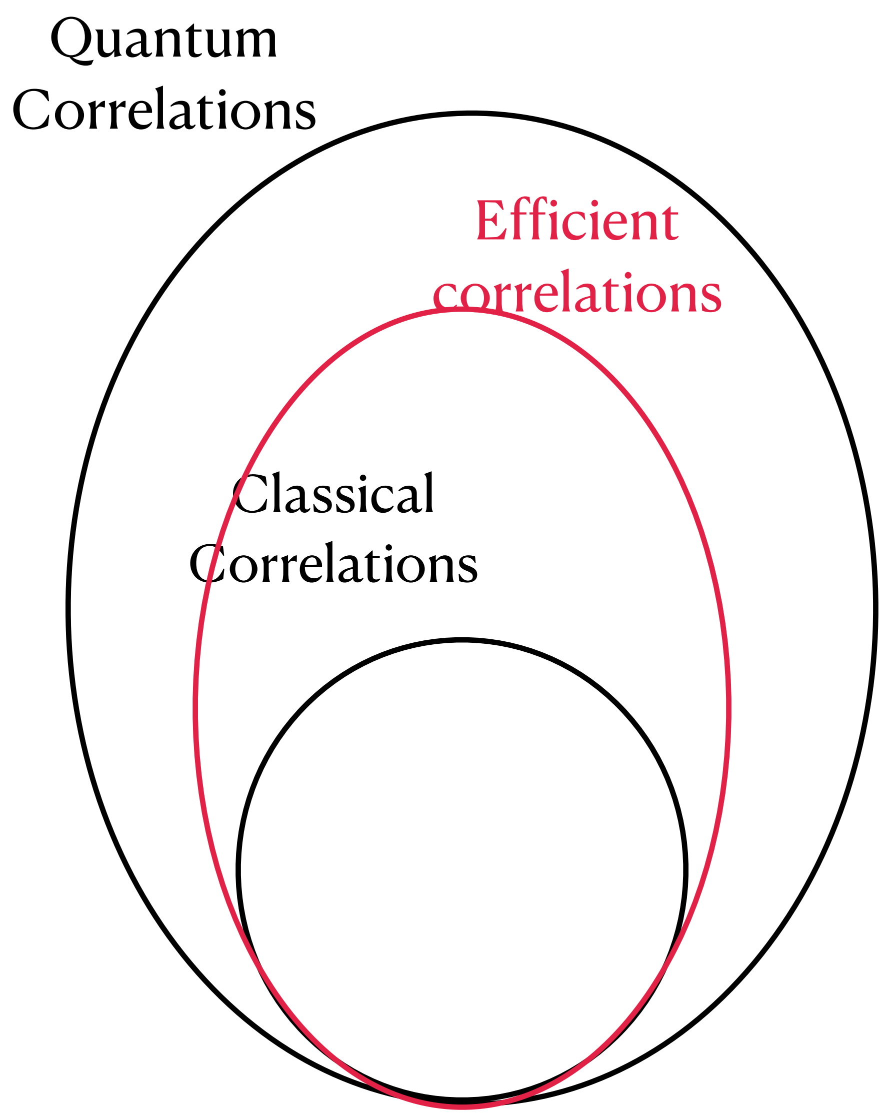


- Restricting the dimension of observable => nondeterministic classes
- Requiring that the observables are efficiently implementable (in BQP)

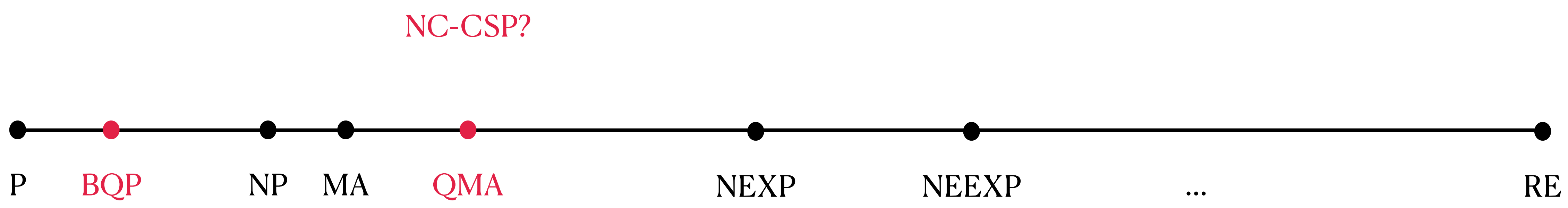
Remember this picture?



Remember this picture?



Open problem



$$\max \sum \frac{1 - tr(X_i X_j)}{2}$$

s.t. X_i is unitary with ± 1 eigenvalues

and X_i has an efficient circuit

- Two generalization of CSPs in quantum information
  - Local Hamiltonians
  - NC-CSPs
- NC-CSPs share the algebraicity of classical CSPs
- We have been able to reach almost the same maturity in NC-CSPs
- Many of the CS tools applicable to CSPs are algebraic in nature
- For Local Hamiltonian we need to invent new tools
- But QMA we may be able to understand better
  - if we find an NC-CSP that captures it!